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A Reduced Order Model for Damping Derived from CFD Based Aeroelastic Simulations

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The prediction of flutter onset based on aerodynamic modelling using CFD can be made using an augmented system of equations. Computational times similar to those required for CFD steady state calculations have been reported for wing test cases. However, for such methods to be fully useful, information about damping must be obtainable without reverting to full order time domain simulation. This paper presents a method for computing damping based on a reduced order modelling approach which systematically derives a two degree of freedom model from the full discrete system of equations. The method is based on a change of variables which employs the critical eigenvector of the aeroelastic system. The ability of this model to predict the damping for a model problem and two wing test cases is shown.

I. Introduction

Computational aeroelasticity has developed rapidly, with attention focussing on time marching calculations using CFD, where the response of a system to an initial perturbation is calculated to determine growth or decay, and from this to infer stability. Recent and impressive example calculations have been made for complete aircraft configurations (see[1,2,3] amongst others).

The time domain method is powerful because of its generality and ease of use. However, basing an investigation of system dynamics in the time domain has one major drawback, namely the computational cost. This has led to an intensive effort to extract the useful information out of the full CFD model of the aerodynamics to provide a cheaper model which still retains the essential physics of the problem. Examples include proper orthognal decomposition[4] which involves the extraction of modes using a limited set of time snapshots of the flow evolution, a Volterra series which relates the aerodynamic response to some input by a kernel[4] and system identification where a linear model is calculated from a limited time evolution of the aerodynamic response to some input. To date no single method has proved its utility on general aeroelastic problems.

Recent work has built on the ideas first presented by Morton and Beran in reference[5] to calculate the onset of flutter through a Hopf Bifurcation. This is achieved through the solution of a modified system of equations at a cost comparable to a steady state CFD solution, giving a considerable advantage compared with using unsteady calculations to bracket the flutter speed. Successful application of the method was made for aerofoils in reference[9] and for wings in reference.[10]

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Whilst knowledge of the onset of instability is important, other pieces of information are required in practice. For example, flight tests measure damping and compare this with predictions to inform decisions about future test points. If the stability boundary is to be crossed in flight then knowledge of the limit cycle oscillation amplitude is required. If this requires recourse to time domain simulations then much of the advantage of the methods of the previous paragraph is lost. A systematic approach to model reduction is therefore required to supplement the rapid prediction of the flutter point.

The perturbation method of multiple scales can be applied to determine behaviour close to a bifurcation. This approach was used in[6] for a two dimensional aeroelastic problem and was presented in[7] for the Duffing oscillator and in[8] for a three equation model problem. The idea is use a perturbation parameter ϵ to separate out events at different time scales. The terms of $O(\epsilon)$ provide a description of the linear dynamics of the system, and higher order terms provide information about the effect of nonlinearities.

The centre manifold theory provides a route to reducing a large dimension model to its essential dynamics. However, the practical obstacles to applying this approach to full order systems of more than 10 degrees of freedom have proved formidable. The current paper addresses this problem by using a change of coordinates for the system variables which allows the reduced model to be derived whilst preserving most of the structure of the original system, making manipulation of the reduction practical. Information from the direct solution of the bifurcation point and its associated eigenvalue and eigenvectors[9,10] is used in this change of variables and the subsequent manipulation.

The current paper adopts this approach to calculating a reduced order model. Further simplification of the approach is made for the prediction of damping and the method is assessed for a model problem followed by application to two flexible wings. The full centre manifold reduction is described in the appendix and is available for the model problem because the second and third Jacobian matrices of the discrete operator, required for the full centre manifold method but not for the simplified damping method, have been calculated analytically. This task is much more difficult for the aeroelastic operator and so only the damping model has been used for the wings. The paper continues with the formulation of the reduced modelling, followed by a description of the full order systems. Results are then presented and evaluated, followed by conclusions.

II. Formulation

A. Stability Calculation

Consider the nonlinear system of equations

$$\dot{x} = f(x, \mu), \quad x \in \Re^n \tag{1}$$

An equilibrium $f(x_0, \mu) = 0$ experiences a loss of stability through a Hopf bifurcation for values of μ such that $\partial f/\partial x = A(x_0, \mu)$ has a pair of eigenvalues $\pm i\omega$ which cross the imaginary axis. Denoting the corresponding eigenvector by $q = q_1 + iq_2$, the behaviour of the critical eigenpair ω and q can be written as

$$Aq = i\omega q. \tag{2}$$

This equation can be written in terms of real and imaginary parts as $Aq_1 + \omega q_2 = 0$ and $Aq_2 - \omega q_1 = 0$. A unique eigenvector is chosen by scaling against a constant real vector q_s to produce a chosen complex value, taken to be $0 + 1i$. This yields two additional scalar equations $q_s^T q_1 = 0$ and $q_s^T q_2 - 1 = 0$.

A bifurcation point can be calculated directly by solving the augmented system of equations

$$f_A(x_A) = 0 \tag{3}$$

where

$$f_A = \begin{bmatrix} f \\ Aq_1 + \omega q_2 \\ Aq_2 - \omega q_1 \\ q_s^T q_1 \\ q_s^T q_2 - 1 \end{bmatrix} \tag{4}$$

and $x_A = [x, q_1, q_2, \mu, \omega]^T$.

The bifurcation point can be calculated through a solution of equation 4 using Newton's method. This has been achieved for aerofoils free to move in pitch and plunge[9] and for flexible wings.[10] Full details of the calculation method are given in those references.

B. Damping Calculation

The eigenvector which goes critical at a Hopf bifurcation will also be the least lightly damped mode for parameter values in a region below the bifurcation value. In this region the aymptotic damping value will be determined by this mode. It is possible to reduce the full system by a change of variables to calculate the damping by only considering a low dimension reduced model. For aeroelastic systems we are dealing with systems of large dimension, and it is advantageous to use a change of variable which involves manipulating the system in its original form as far as possible. A summary of the general formulation based on the centre manifold theory is given in the appendix. In this section a simpler version of the general theory is given to allow damping to be calculated.

The full system can be transformed by using only the vectors corresponding to the critical eigenvalues of A and its transpose A^T. These are calculated from equation 4. The system is projected onto its critical eigenspace and complement. Suppose we have a Taylor expansion of the residual function f about the equilibrium solution x_0 and parameter at the bifurcation point μ_0, giving

$$\dot{\bar{x}} = A\bar{x} + F(\bar{x}, \bar{\mu}), \quad x \in \Re^n \tag{5}$$

where $F(\bar{x}, \bar{\mu})$ has at least quadratic terms and $\bar{x} = x - x_0$, $\bar{\mu} = \mu - \mu_0$. The matrix A has a pair of complex eigenvalues on the imaginary axis $\lambda_{1,2} = i\omega$, $\omega > 0$. Let q be the right eigenvector corresponding to λ_1. Then $\bar{q}$ is the right eigenvector corresponding to λ_2 and

$$Aq = i\omega q, \qquad A\bar{q} = -i\omega\bar{q}$$

The left eigenvector p has the same property

$$A^T p = -i\omega p, \qquad A^T\bar{p} = i\omega\bar{p}.$$

These can be normalised such that $\langle p, q\rangle = 1$ where $\langle p, q\rangle = \sum_{i=1}^{n} \bar{p}_i q_i$. The eigenspace S corresponding to $\pm i\omega$ is two dimensional and is spanned by $\{\Re q, \Im q\}$. The eigenspace T corresponds to all the other eigenvalues of A and is $n-2$ dimensional. Then $y \in T$ if and only if $\langle p, y\rangle = 0$. Since $y \in \Re^n$ while p is complex then two real constraints on y exist and hence it is possible to decompose any $\bar{x} \in \Re^n$ as

$$\bar{x} = zq + \bar{z}\bar{q} + y$$

where $z \in C^1$, $zq + \bar{z}\bar{q} \in S$, and $y \in T$. The complex variable z is a coordinate of S so

$$\begin{cases} z &= \langle p, \bar{x}\rangle \\ y &= \bar{x} - \langle p, \bar{x}\rangle q - \langle \bar{p}, \bar{x}\rangle\bar{q} \end{cases}$$

since $\langle p, \bar{q}\rangle = 0$. The equation (5) then has the form

$$\begin{cases} \dot{z} &= i\omega z + \langle p, F(zq + \bar{z}\bar{q} + y, \bar{\mu})\rangle \\ \dot{y} &= Ay + F(zq + \bar{z}\bar{q} + y, \bar{\mu}) - \langle p, F(zq + \bar{z}\bar{q} + y, \bar{\mu})\rangle q - \langle p, F(zq + \bar{z}\bar{q} + y, \bar{\mu})\rangle\bar{q} \end{cases}$$

This system is $(n+2)$ dimensional but we have two constraints on y.

Now, in general at this stage a centre manifold reduction should be used to obtain a relationship between y and z which allows the critical dynamics to be calculated from the z equation only. This treatment allows nonlinear features such as Limit Cycle Oscillations to be calculated from the reduced model. A description of the method to perform this reduction is given in appendix A, and is referred to in the results section as the *centre manifold reduced model*. Here we are interested in calculating damping for parameter values below the bifurcation point. In this case the influence of the component y from the non-critical space is damped faster than the critical component z. We therefore neglect the influence of y altogether which removes the need for the centre manifold reduction. Further justification for this approximation will be given from results for a test problem below.

The damping is therefore determined by solving the equation

$$\dot{z} \quad = \quad i\omega z + \langle p, F(zq + \bar{z}\bar{q}, \bar{\mu})\rangle \ .$$

This system is two dimensional. Finally, we need to calculate the form of F. Expanding the function f in a Taylor series about the equilibrium solution x_0 and parameter μ_0 gives

$$\begin{aligned} f(\bar{x}, \bar{\mu}) \quad = \quad & f(x_0, \mu_0) + \frac{\partial f}{\partial x}\bar{x} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\bar{x}\bar{x} + \frac{1}{6}\frac{\partial^3 f}{\partial x^3}\bar{x}\bar{x}\bar{x} + \\ & \frac{\partial f}{\partial \mu}\bar{\mu} + \frac{1}{2}\frac{\partial^2 f}{\partial \mu^2}\bar{\mu}^2 + \frac{1}{6}\frac{\partial^3 f}{\partial \mu^3}\bar{\mu}^3 + \\ & \frac{\partial^2 f}{\partial \mu \partial x}\bar{\mu}\bar{x} + \frac{1}{2}\frac{\partial^3 f}{\partial \mu^2 \partial x}\bar{\mu}^2\bar{x} + \frac{1}{2}\frac{\partial^3 f}{\partial \mu \partial x^2}\bar{\mu}\bar{x}\bar{x} + ... \end{aligned}$$

where all derivatives are evaluated at (x_0, μ_0). We can simplify this by noting that $f(x_0, \mu_0) = 0$ and neglecting terms which are quadratic and higher in $\bar{x}$ and $\bar{\mu}$. This leaves

$$f(\bar{x}, \bar{\mu}) \approx A\bar{x} + \frac{\partial f}{\partial \mu}\bar{\mu} + \frac{\partial A}{\partial \mu}\bar{\mu}\bar{x}$$

which means that

$$F(\bar{x}, \bar{\mu}) = f_\mu \bar{\mu} + A_\mu \bar{\mu}\bar{x} \tag{6}$$

and hence

$$\langle p, F(zq + \bar{z}\bar{q}, \bar{\mu})\rangle = \langle p, f_\mu \bar{\mu}\rangle + \langle p, A_\mu \bar{\mu}\bar{x}\rangle.$$

Using the change of coordinates and pulling the values of z and $\bar{z}$ through the inner product we obtain

$$\langle p, \bar{\mu}A_\mu \bar{x}\rangle = z\langle p, \bar{\mu}A_\mu q\rangle + \bar{z}\langle p, \bar{\mu}A_\mu \bar{q}\rangle.$$

This allows the reduced model to be written as a constant coefficient two degree of freedom system.

This model is referred to in the results section as the *damping reduced model*. If the term F is neglected altogether then we only retain linear terms, and this is referred to as the linear reduced model. Further simplification of the damping reduced model is possible for the aeroelastic system and will be discussed below.

To summarise, the damping calculation proceeds in the following steps:

1. using the direct solver, calculate the Hopf bifurcation point and the critical eigenvalue and eigenvalue and the corresponding eigenvector of A^T
2. calculate the projected two degree of freedom model using

$$\dot{z} = i\omega z + z\langle p, \bar{\mu}A_\mu q\rangle + \bar{z}\langle p, \bar{\mu}A_\mu \bar{q}\rangle$$

3. use the two degree of freedom model to compute the response of z to an initial disturbance for values of $\mu < \mu_0$. This solution can be transformed back to the original variables using

$$\bar{x} = zq + \bar{z}\bar{q}$$

C. 2D non-adiabatic tubular reactor with axial mixing

To test the solution methodology for the augmented system, a model problem is considered which describes the unsteady behaviour of a non-adiabatic tubular reactor with axial mixing,[1112]

$$\begin{aligned} \frac{\partial y}{\partial t} \quad &= \quad \frac{1}{Pe_m}\frac{\partial^2 y}{\partial x^2} - \frac{\partial y}{\partial x} - \mu y \exp\left(\Gamma - \frac{\Gamma}{\Theta}\right) \\ \frac{\partial \Theta}{\partial t} \quad &= \quad \frac{1}{Pe_h}\frac{\partial^2 \Theta}{\partial x^2} - \frac{\partial \Theta}{\partial x} - \beta(\Theta - \bar{\Theta}) + \mu\alpha y \exp\left(\Gamma - \frac{\Gamma}{\Theta}\right) \end{aligned} \tag{7}$$

where Pe_m, Pe_h, β, α, Γ, and $\bar{\Theta}$ are fixed constants and μ is the bifurcation parameter. The boundary conditions $(t > 0)$ are given by

$$\frac{\partial y}{\partial x} = Pe_m(y - 1) \quad \frac{\partial \Theta}{\partial x} = Pe_m(\Theta - 1) \quad (x = 0)$$

$$\frac{\partial y}{\partial x} = \frac{\partial \Theta}{\partial x} = 0 \quad (x = 1)$$

For the results presented here the constants are set to $Pe_m = 5$, $Pe_h = 5$, $\beta = 2.5$, $\alpha = 0.5$, $\Gamma = 25$, and $\bar{\Theta} = 1.0$.

The system is discretised using a cell centred finite difference scheme so that the first and second differences are approximated by

$$\left.\frac{\partial^2 y}{\partial x^2}\right|_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} \qquad \left.\frac{\partial y}{\partial x}\right|_i = \frac{y_{i+1} - y_{i-1}}{2h}.$$

Here a uniform mesh of spacing h is used with the i-th point at $x_i = ih$ for $(i = 0, \ldots, n)$. The boundary conditions for $x = 1$ are applied by setting halo cell values to be identical to the values in the adjacent interior cell.

The solution for an equilibrium and also of the augmented system is by the full Newton method with the use of the exact Jacobian on the left hand side. For the augmented system, and the various types of reduced model, the first and second order Jacobian terms have been calculated analytically and were checked using finite differences. To check the reduced results, unsteady time stepping is also considered. An explicit method is used which results in a large number of time steps ($\Delta t = 1/500$ is required for stability). The bifurcation point is bracketed between a steady solution at one parameter value and an unsteady solution at a second value. Each new calculation halves the length of the region bracketing the bifurcation value.

D. 3D Aeroelastic System Based on the Euler Equations

The three-dimensional Euler equations can be written in conservative form and Cartesian coordinates as

$$\frac{\partial \mathbf{w}_f}{\partial t} + \frac{\partial \mathbf{F}^i}{\partial x} + \frac{\partial \mathbf{G}^i}{\partial y} + \frac{\partial \mathbf{H}^i}{\partial z} = 0 \tag{8}$$

where $\mathbf{w}_f = (\rho, \rho u, \rho v, \rho w, \rho E)^T$ denotes the vector of conserved variables. The flux vectors $\mathbf{F}^i$, $\mathbf{G}^i$ and $\mathbf{H}^i$ are,

$$\mathbf{F}^i = \begin{pmatrix} \rho U^* \\ \rho u U^* + p \\ \rho v U^* \\ \rho w U^* \\ U^*(\rho E + p) + \dot{x} \end{pmatrix}, \tag{9}$$

$$\mathbf{G}^i = \begin{pmatrix} \rho V^* \\ \rho u V^* \\ \rho v V^* + p \\ \rho w V^* \\ V^*(\rho E + p) + \dot{y} \end{pmatrix} \mathbf{H}^i = \begin{pmatrix} \rho W^* \\ \rho u W^* \\ \rho v W^* + p \\ \rho w W^* + p \\ W^*(\rho E + p) + \dot{z} \end{pmatrix}. \tag{10}$$

In the above ρ, u, v, w p and E denote the density, the three Cartesian components of the velocity, the pressure and the specific total energy respectively, and U^*, V^*, W^* the three Cartesian components of the velocity relative to the moving coordinate system which has local velocity components $\dot{x}$, $\dot{y}$ and $\dot{z}$, i.e.

$$U^* = u - \dot{x} \tag{11}$$

$$V^* = v - \dot{y} \tag{12}$$

$$W^* = w - \dot{z} \tag{13}$$

The wing deflections $\delta \mathbf{x}_s$ are defined at a set of points $\mathbf{x}_s$ by

$$\delta \mathbf{x}_s = \Sigma \alpha_i \phi_i \tag{14}$$

where ϕ_i are the mode shapes calculated from a full finite element model of the structure and α_i are the generalised coordinates. By projecting the finite element equations onto the mode shapes the scalar equations

$$\frac{d^2 \alpha_i}{dt^2} + \omega_i^2 \alpha_i = \mu \phi_i^T \mathbf{f}_s \tag{15}$$

are obtained where $\mathbf{f}_s$ is the vector of aerodynamic forces at the structural grid points, and $\mu = c^5 \rho_\infty U_\infty^2$. These equations are rewritten as a system in the form

$$\frac{d\mathbf{w}_s}{dt} = \mathbf{R}_s \tag{16}$$

where $\mathbf{w}_s = (....., \alpha_i, \dot{\alpha}_i, ...)^T$ and $\mathbf{R}_s = (....., \dot{\alpha}_i, \mu\phi_i^T \mathbf{f}_s - \omega_i^2 \alpha_i, ...)^T$.

The aerodynamic forces are calculated at face centres on the aerodynamic surface grid and these must be transfered to the structural grid.. This problem was considered in references[13] and,[14] where a method was developed, called the constant volume tetrahedron (CVT) transformation. Denoting the fluid grid locations and aerodynamic forces as $\mathbf{x}_a$ and $\mathbf{f}_a$, then

$$\delta\mathbf{x}_a = \mathcal{S}(\mathbf{x}_a, \mathbf{x}_s, \delta\mathbf{x}_s)$$

where $\mathcal{S}$ denotes the relationship defined by CVT. In practice this equation is linearised to give

$$\delta\mathbf{x}_a = S(\mathbf{x}_a, \mathbf{x}_s)\delta\mathbf{x}_s$$

and then by the principle of virtual work, $\mathbf{f}_s = S^T \mathbf{f}_a$.

The grid speeds on the wing surface are also needed and these are approximated directly from the linearised transformation as

$$\delta\dot{\mathbf{x}}_a = S(\mathbf{x}_a, \mathbf{x}_s)\delta\dot{\mathbf{x}}_s$$

where the structural grid speeds are given by

$$\delta\dot{\mathbf{x}}_s = \Sigma\dot{\alpha}_i\phi_i. \tag{17}$$

We have to deal with the deforming geometry. This is achieved using Transfinite Interpolation of Displacements (TFI) within the blocks containing the wing. The wing surface deflections are interpolated to the volume grid points $\mathbf{x}_{ijk}$ as

$$\delta\mathbf{x}_{ijk} = \psi_j^0 \delta\mathbf{x}_{a,ik} \tag{18}$$

where ψ_j^0 are values of a blending function[15] which varies between one at the wing surface (here j=1) and zero at the block face opposite. The surface deflections $\mathbf{x}_{a,ik}$ are obtained from the transformation of the deflections on the structural grid and so ultimately depend on the values of α_i. The grid speeds can be obtained by differentiating equation (18) to obtain their explicit dependence on the values of $\dot{\alpha}_i$.

We will consider the bifurcation parameter as μ. For symmetric wings at zero incidence any equilibrium solution has the wing undeflected. This means that $\mathbf{f}_s = 0$ at the equilibrium solution. The fluid equations do not depend explicitly on μ. Therefore $f_\mu = 0$ in equation 4 for an equilibrium solution. Also, the linear dependence of the structural equations on μ means that A_μ is constant and only has non zero terms in the small number of rows corresponding to the structural equations.

III. Results

A. Tubular Reactor

The rich solution space for this model problem is shown in figure 1. This includes stable and unstable equilibria, limit points and Hopf bifurcation points. There is also a hysteresis loop for increasing and decreasing μ. The solution is characterised by the maximum value of Θ within the domain. The equilibrium solutions for varying μ are shown in figure 1. For $\mu < 0.165$ and $\mu > 0.180$ this equilibrium is stable and the solution to equation (7) is steady. For $0.165 < \mu < 0.180$ the equilibrium is unstable and a limit cycle oscillation is formed. Depending on whether the parameter μ is increased (solid line) or decreased (solid switching to dashed lines with increasing μ) a different equilibrium is obtained, indicating hysteresis. The equilibria were mapped out using a continuation method with Newton's method for the corrector stage. In addition, time marching calculations were done to map out the stability of these equilibria.

Next, the augmented system (equation 3) was solved to find the bifurcation points. If the initial guess is poor then the solution diverges. For the current calculations the following initial guess was used: $\mu = 0.16$, $x_{2i} = 1.0$, $x_{2i+1} = 0.0$, $P_{1_i} = \sqrt{n1}$, $P_{2_{2i}} = \sqrt{n1}$,

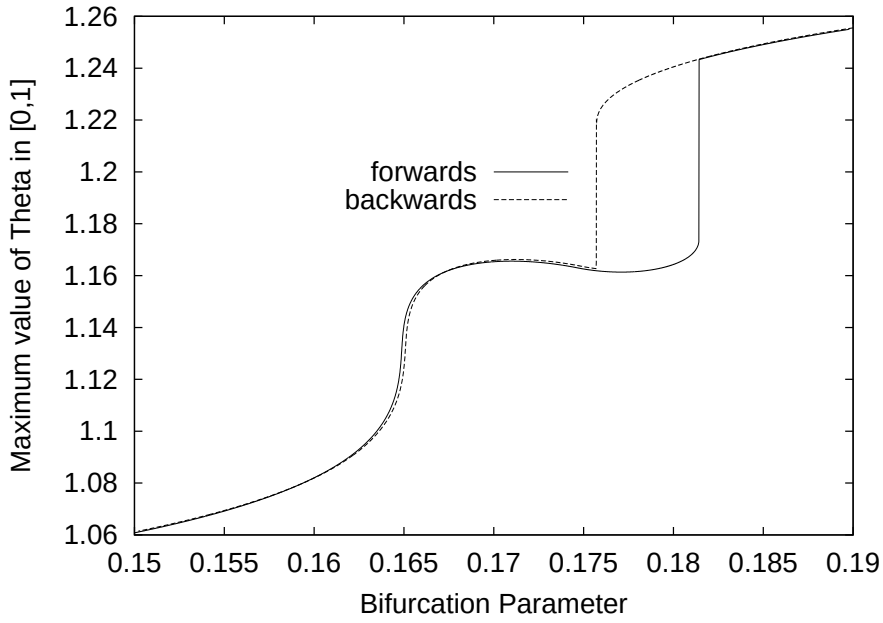


Figure 1. The equilibrium solution as mapped out by a continuation method varing the bifurcation parameter μ

$P_{2_{2i+1}} = -\sqrt{n1}$, $q = P_2$ and the eigenvalue i. By changing the initial conditions the Newton iterations can be made to converge to the second Hopf point at $\mu = 0.180$. Starting from this guess the iterations had to be under-relaxed by a factor 0.5 until the domain of quadratic convergence was reached (the criteria used was based on the initial residual being reduced by half). A sequence of grids was used to show mesh independence and a second method of initialisation was used by taking the final solution from the previous grid in the sequence as the starting solution on the next grid. No relaxation was required using this technique.

Damping calculations were made using the reduced model. Various results are compared to gain some insight into the behaviour of the different options. The benchmark is the time domain solution of the full system, which is indicated by the dots on the time response curves. Secondly, the full centre manifold reduction is based on the Taylor expansion of the residual including third order terms, and using the reduction methodology described in the appendix. Finally, the key results are obtained using the simpler damping reduction using the Tayor expansion of equation 6.

We consider results about the bifurcation point $\mu_0 = 0.16508$ and for values of $\bar{\mu}$ of -0.00005, -0.0001 and -0.0002. Although it is beyond the scope of this paper, results are also shown in figure 2 for $\bar{\mu} = 0.00007$ which results in a limit cycle. The centre manifold reduction predicts the LCO response perfectly.

The damped responses for $\bar{\mu} = -0.00005$, $\bar{\mu} = -0.0001$ and $\bar{\mu} = -0.0002$ are shown in figure 3. The centre manifold results, which arise from solving a two degrees of freedom system, agree perfectly with the full order (here 512 degrees of freedom) system results for all three cases. Finally, the damping reduced model predicts the response very well. Given the relative simplicity of calculating the damping reduced model, these results suggest a strong preference for this approach.

B. AGARD 445.6 Wing

The behaviour of the method is next investigated for the aeroelastic response of the AGARD 445.6 wing. Time domain and bifurcation results are given in.[10] The grid has 17900 points and is optimised to have a large number of points in the tip region which is critical for predicting flutter onset. The four important modes from the structural model, which is of the plate variety, were retained. The flutter boundary is shown in figure 4, with below the curve being the stable region.

Second and Third Jacobians of the full CFD operator proved unreliable from finite

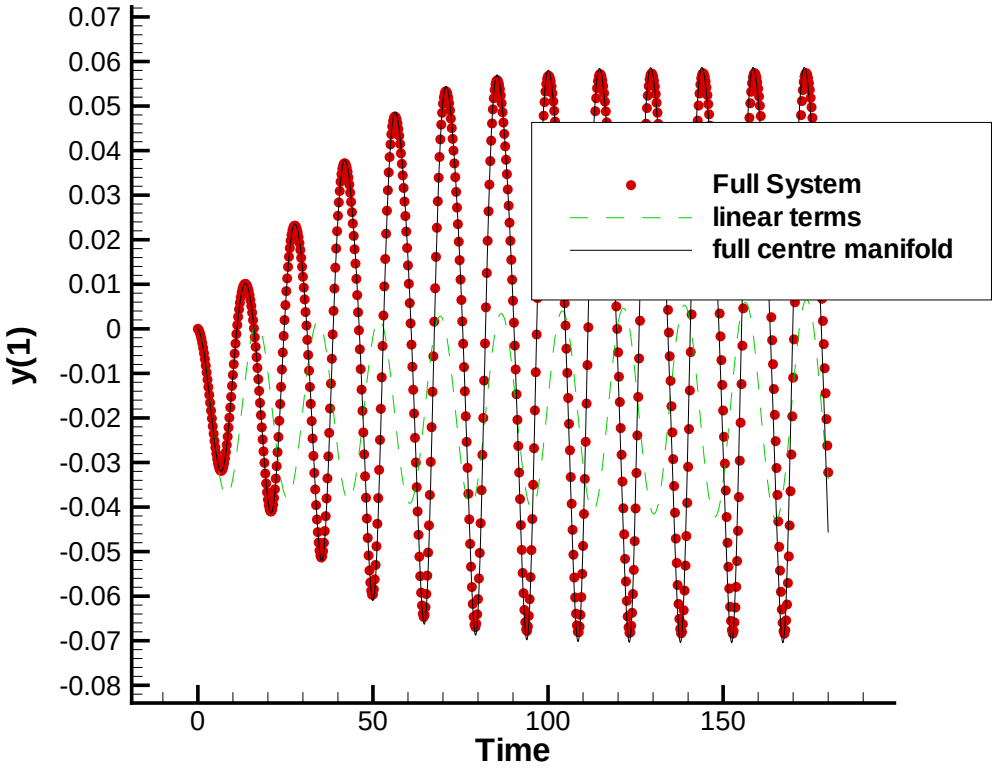
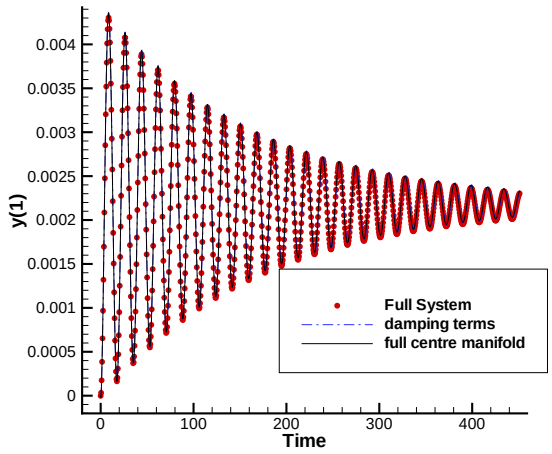
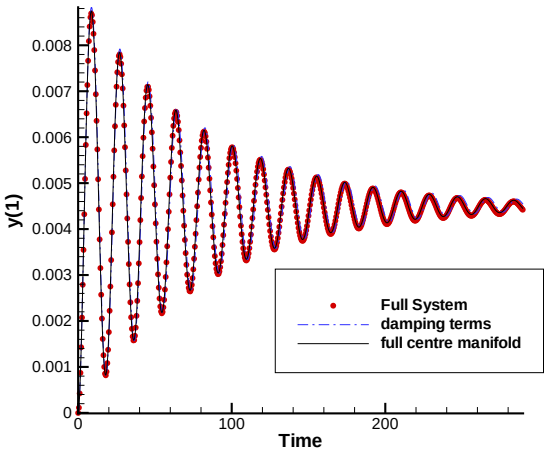


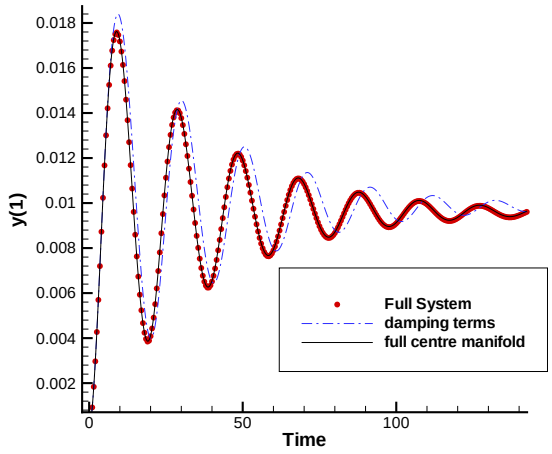
Figure 2. Comparison of results from the full model, centre manifold reduced model and linear reduced model for an LCO response at $\bar{\mu} = 0.00007$



(a) $\bar{\mu} = -0.00005$



(b) $\bar{\mu} = -0.0001$



(c) $\bar{\mu} = -0.0002$

Figure 3. Comparison of results from the full model, centre manifold reduced model, damping reduced model and linear reduced model for damped responses at $\bar{\mu} = -0.00005$, $\bar{\mu} = -0.0001$ and $\bar{\mu} = -0.0002$.

difference calculations and remedies to this were left to future work. Therefore the centre manifold reduced model was unavailable for the aeroelastic cases.

The reduced damping model predictions were calculated for values of dynamic pressure which are 5%,10%,20% and 40% below the bifurcation value for Mach numbers of 0.67, 0.90, 0.96 and 1.07. The reduced model (two degrees of freedom) responses are compared with the full order system (89508 degrees of freedom) in figures 5 - 8. A number of comments can be made. First, as the dynamic pressure tends to the flutter value, the reduced model predictions converge to those of the full model as expected. The more heavily damped results at M=0.67 show more discrepancy than the other three cases, which show good agreement, even at 60 % of the flutter value. It is possible that the addition of higher order terms from the centre manifold reduction may allow improved prediction for the heavily damped cases and this will be investigated in future work. The typical CPU time for a full order time domain calculation is 142 times a steady state calculation. The reduced model takes a negligible time to run, and is formed (once only for each Mach number) at a cost comparable to 3.2 steady state calculations. Once formed the responses can be approximated almost free for any value of the dynamic pressure below the critical value.

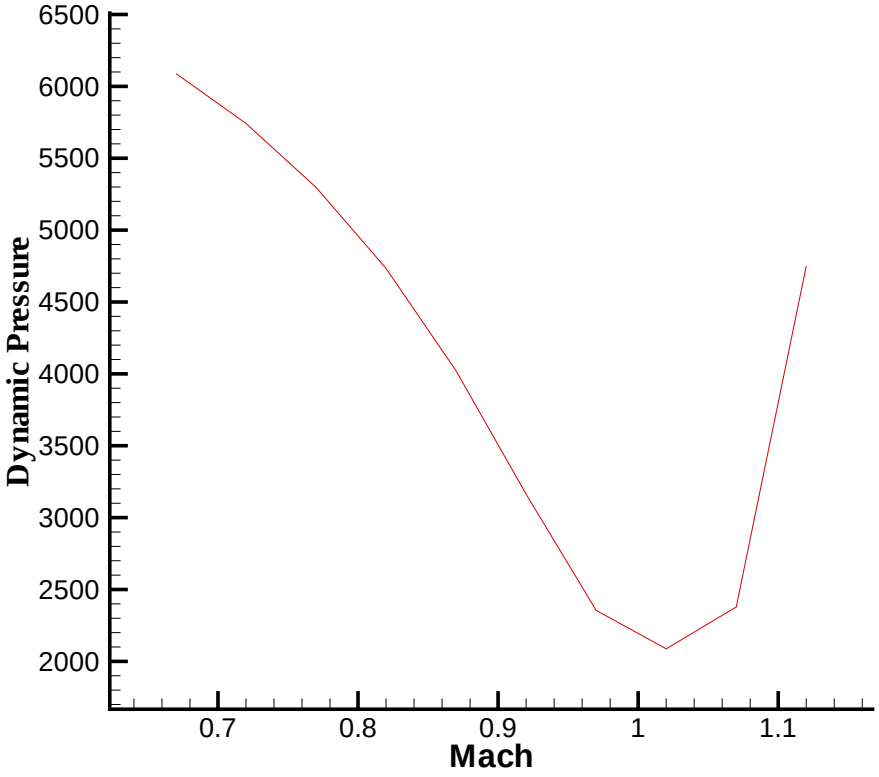
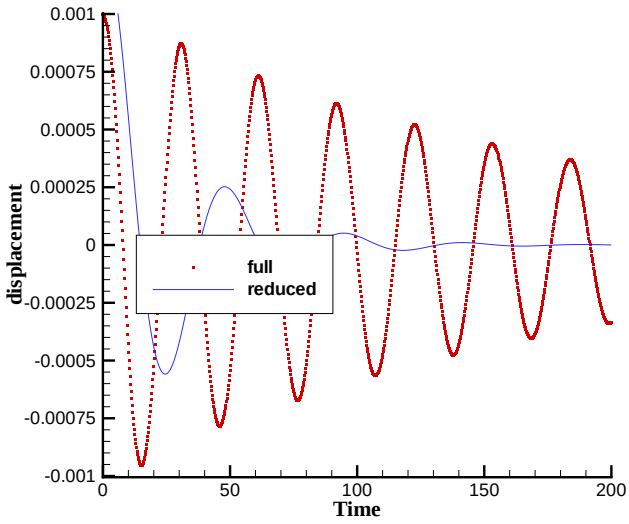


Figure 4. Flutter boundary for AGARD wing traced out using the bifurcation solver.

C. Hawk Wing

Finally, we present results representing the wing of the Hawk trainer which is manufactured by BAE SYSTEMS. A previous study of the flutter characteristics of this aircraft was reported in reference[3] using the time domain approach. Predictions for several models of the Hawk were compared, including a linear method, a CFD based model of the wing-body-tailplane, and the wing alone.

In the current calculations an aerodynamic grid with 16644 points was generated which was found to give reliable results in the previous paper through a grid refinement study.



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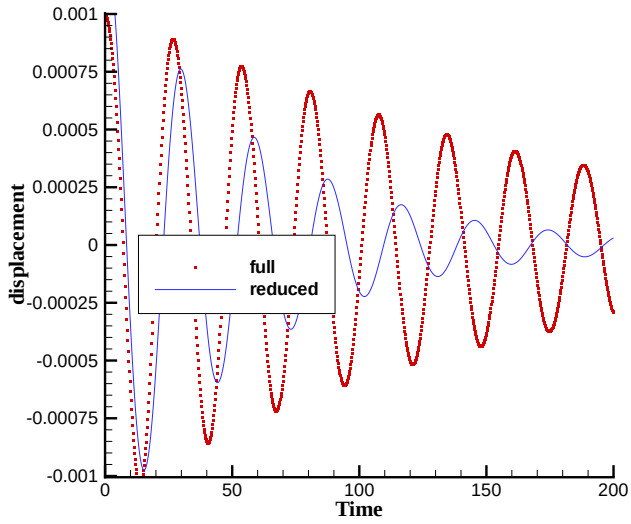

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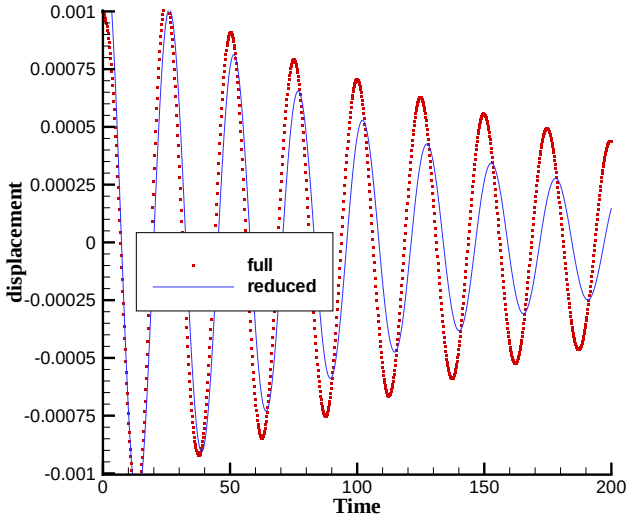

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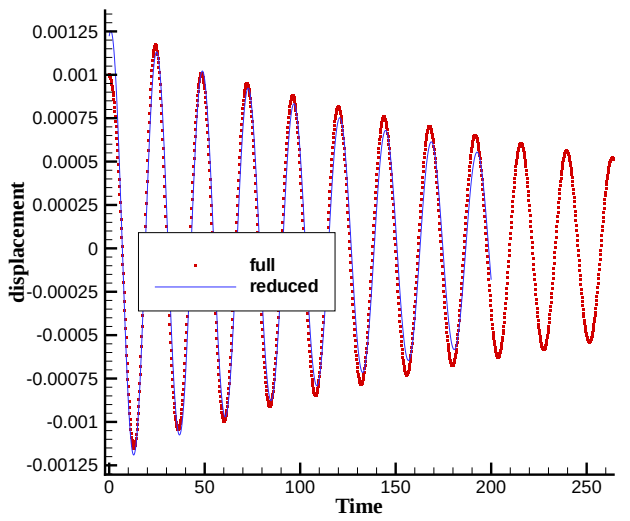

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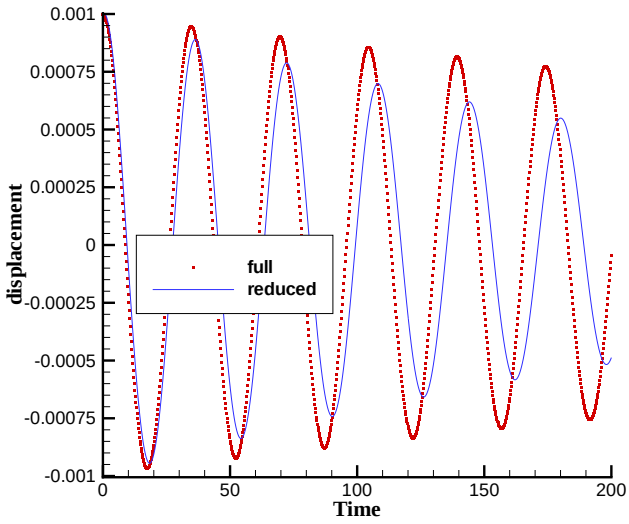
Figure 5. Comparison of results from the full model and damping reduced model at M=0.67 for values of dynamic pressure below the flutter boundary.



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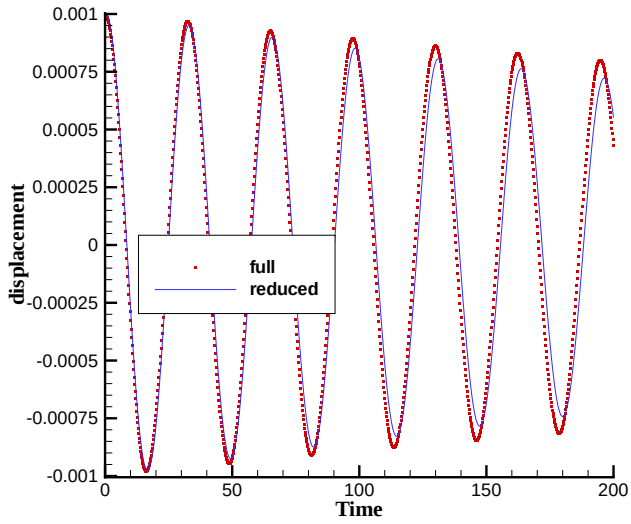

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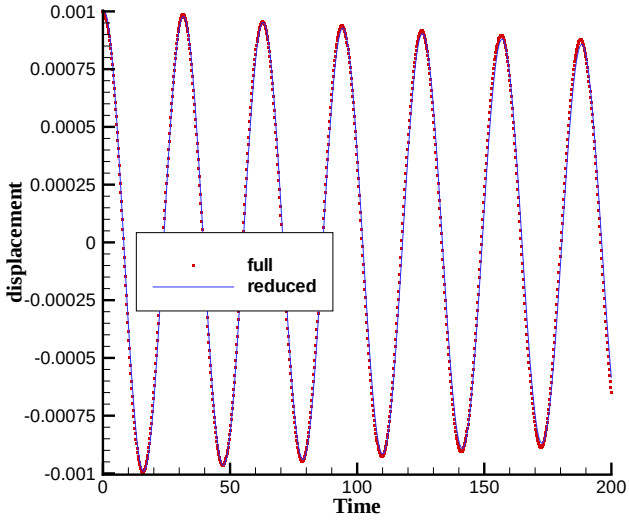

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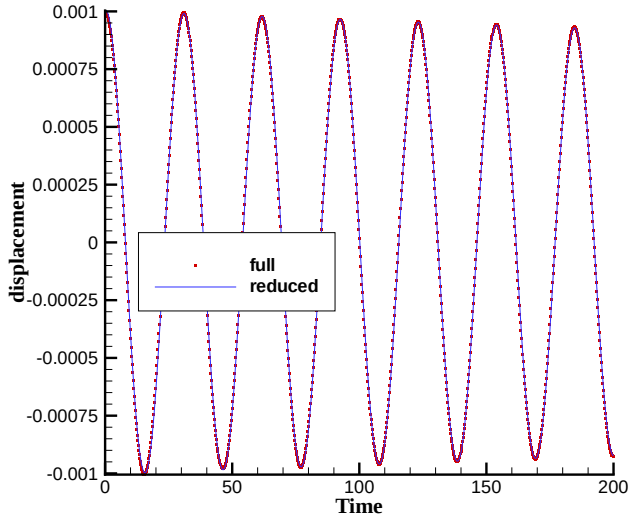
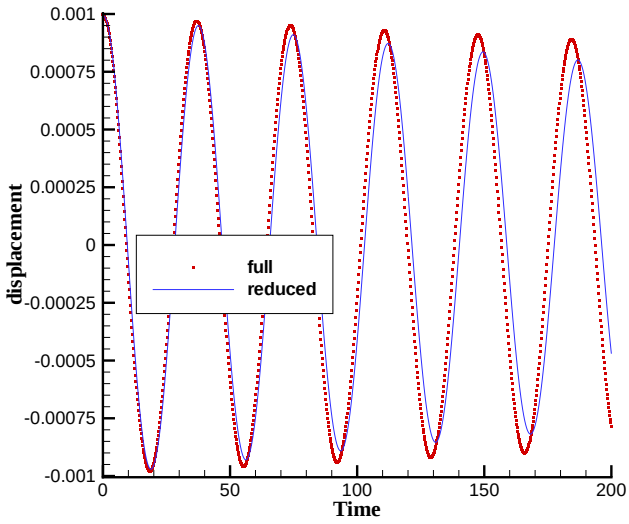


(d) 95%

Figure 6. Comparison of results from the full model and damping reduced model at M=0.90 for values of dynamic pressure below the flutter boundary.



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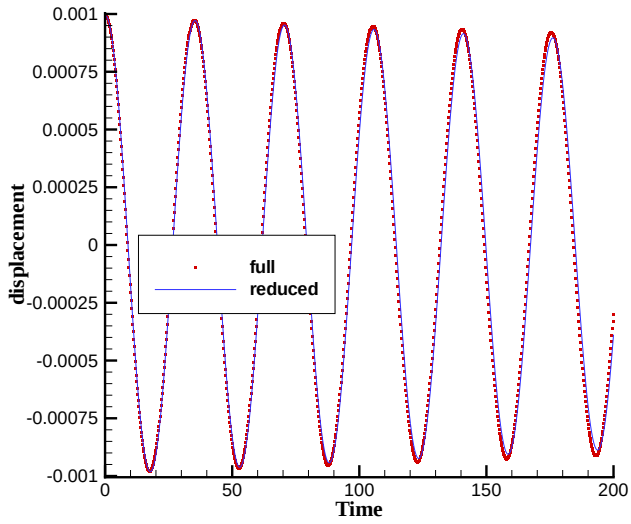

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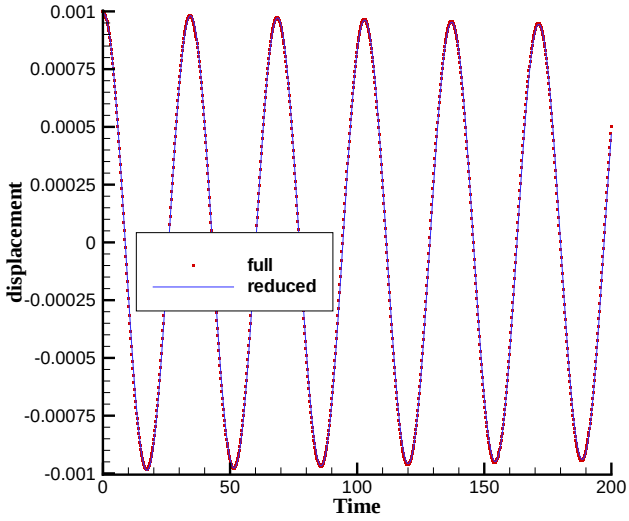

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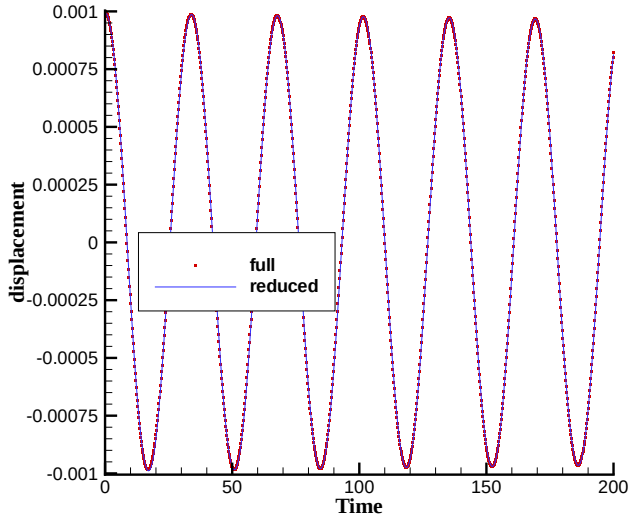

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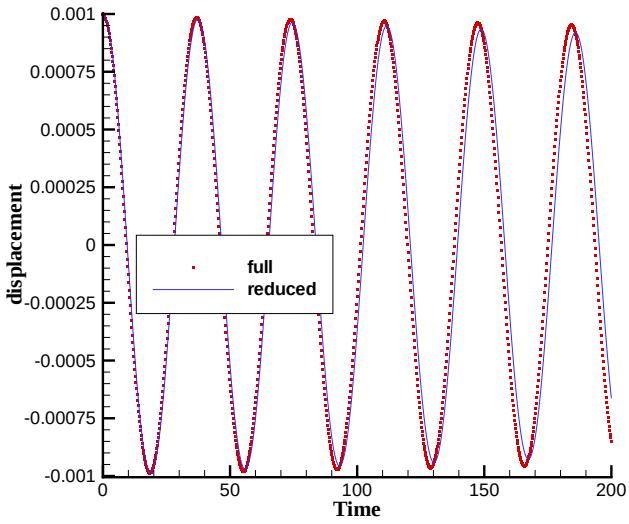
Figure 7. Comparison of results from the full model and damping reduced model at M=0.96 for values of dynamic pressure below the flutter boundary.



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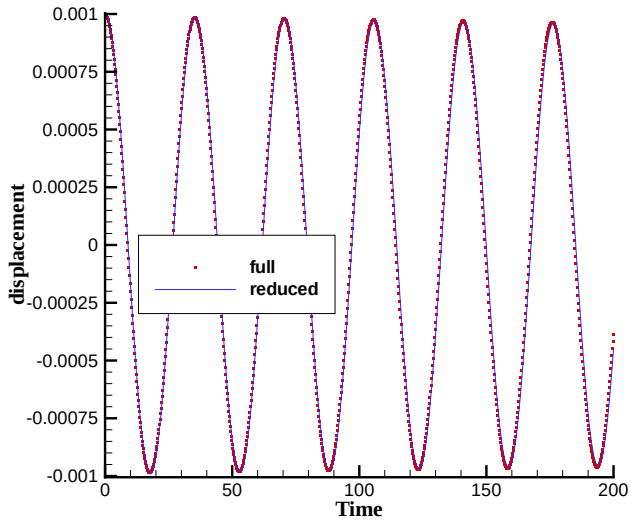

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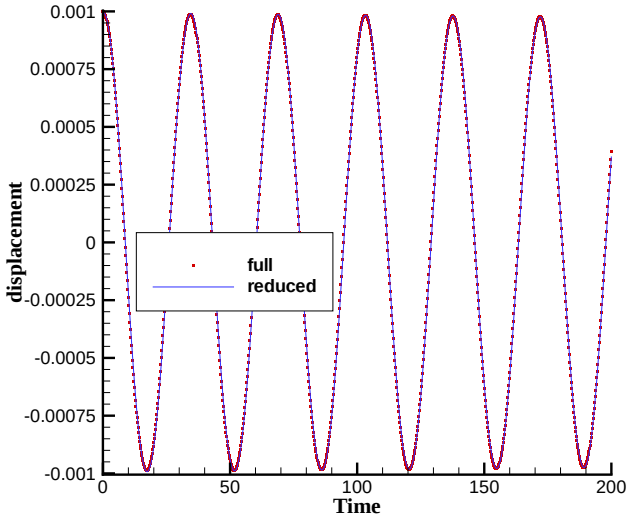

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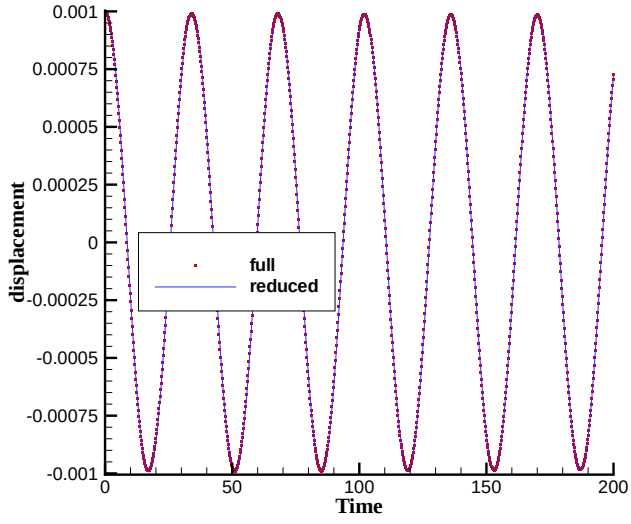


(d) 95%

Figure 8. Comparison of results from the full model and damping reduced model at M=1.07 for values of dynamic pressure below the flutter boundary.

The structural dynamics is represented by a beam model, supplied by BAE SYSTEMS. A full description of how the transformation is done from this structural model is given in reference.[3] The four lowest frequency symmetric non-tailplane modes in the structural model are retained for the flutter calculations. These have frequencies 12.42Hz (1st wing bending), 14.43Hz (the influence of the 1st fuselage bending mode on the wing), 32.46Hz (the influence of the 2nd fuselage bending mode on the wing) and 37.87Hz (the first wing torsion mode). In the linear aeroelastic predictions made with the same four modes, the first wing bending and torsion modes couple, the other two modes having only a small influence on the flutter mechanism. This structural model provided an ideal test case for the CFD based methods. The full aero-structural model has 16652 degrees of freedom.

The bifurcation solver was first used to trace out the flutter boundary which is shown in figure 9. The prediction of damping was then evaluated for the Mach number which results in the lowest point on the transonic flutter dip. Values of dynamic pressure at 98%, 95% and 92% of the critical value were chosen. Even at 98% of the critical value the damping is heavy, in contrast with the AGARD wing for the Mach number in the flutter dip.

The comparison between the damping reduced model and the full order model is shown in figure 10. The agreement at 98 % of the critical value is close and in terms of damping the reduced model predictions are not too disimilar at 95% and 92% either. However, close inspection of the comparisons shows that the frequencies of the responses at the lower two values are significantly in disagreement. Calculation of the eigenspectrum by the inverse power method shows that the closest mode to the imaginary axis is the critical mode only after the bifurcation parameter is 96% of the critical value. Below this a lower frequency mode dominates the response.

The time domain calculation of the full order system in this cases took in excess of 10 hours on a Pentium 4 processor. The reduced model took less than 20 seconds to compute the same case.

IV. Conclusions

Previous work has shown that the flutter onset speed can be computed for CFD-CSD coupled models using fast methods based on the behaviour of the critical eigenvalue. Flutter speeds can be computed in roughly the cost of a steady state calculation, avoiding the large costs associated with time domain analysis.

In the current paper a method which uses information available about the critical eigenvector of the system has been presented which forms a two-degree-of-freedom model to compute the damped response at values of the dynamic pressure below the critical value. This information has the potential to allow rapid evaluation of damping characteristics prior to flight testing.

Results were shown for a Tubular Reactor model problem and then for the aeroelastic behaviour of the AGARD and Hawk wings. Close to the critical parameter value the full order system response is reproduced well by the two degree of freedom model. As the response becomes more heavily damped the agreement becomes less good, but these cases are also less critical in the real situation.

Results using a centre manifold correction were presented for the Tubular Reactor model problem and excellent agreement was obtained with the full order system, even for heavily damped conditions. Further work is underway to accurately evaluate the second and third Jacobians of the aeroelastic operator to allow this correction to be applied for the aeroelastic cases. In addition, and most interestingly, this will allow post-bifurcation behaviour (i.e. LCO's) to be computed also.

V. Acknowledgements

This work was supported by BAE SYSTEMS, Engineering and Physical Sciences Reserach Council and Ministry of Defence, and is part of the work programme of the Partnership for Unsteady Methods in Aerodynamics (PUMA) Defence and Aerospace Research Partnership (DARP).

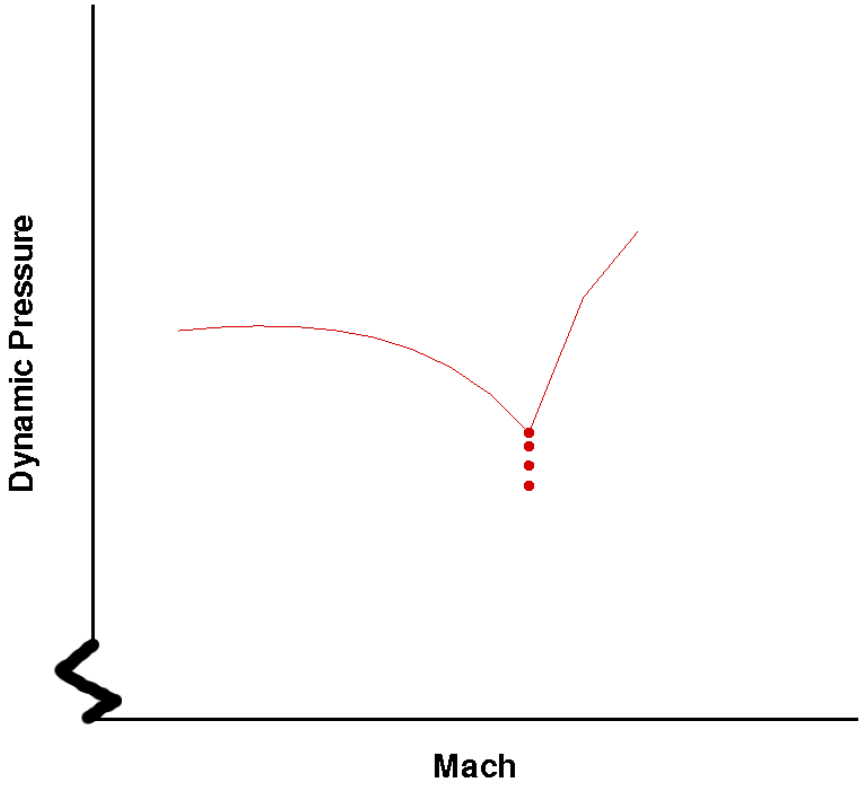
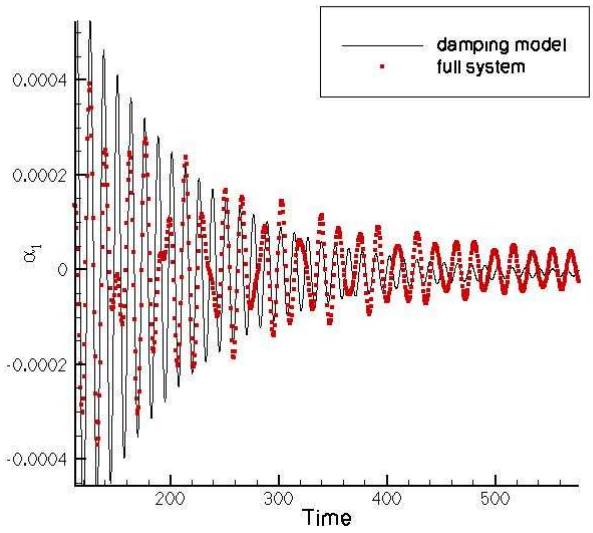
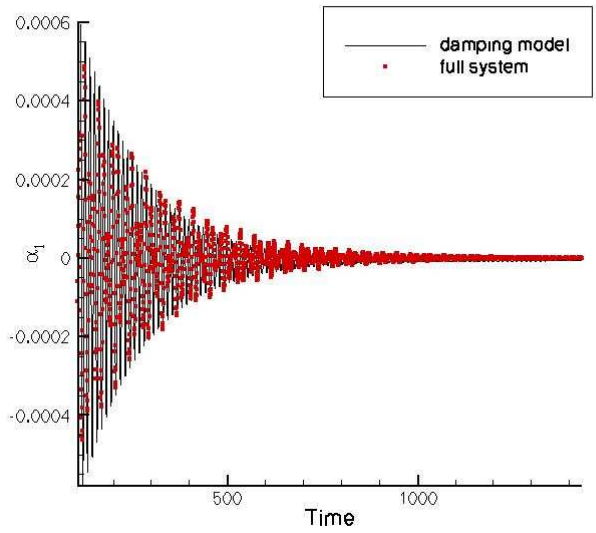


Figure 9. Flutter boundary for Hawk wing traced out using the bifurcation solver. The values of dynamic pressure at which the damping model is compared with full order results are indicated by the dots.



(a) 92%



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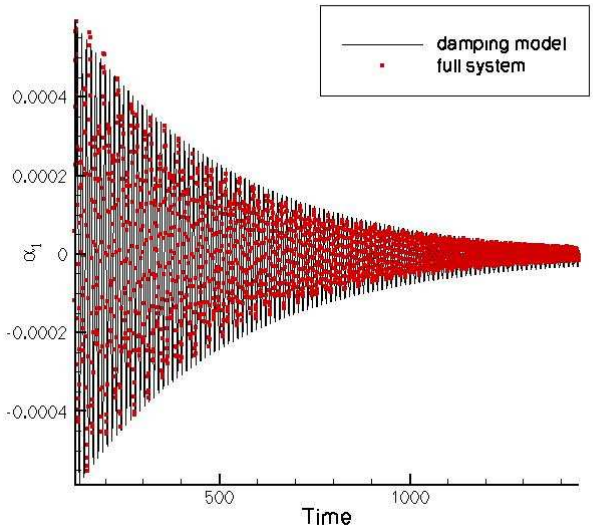

(c) 98%

Figure 10. Comparison of results from the full model and damping reduced model for the Hawk wing in the transonic dip and for values of dynamic pressure below the flutter boundary.

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A. Classical Model Reduction

Consider the nonlinear system of equations

$$\dot{x} = f(x), \quad x \in \Re^n \tag{19}$$

where f is sufficiently smooth. We assume that we are at a Hopf bifurcation and hence the Jacobian matrix $\partial f/\partial x$ has 2 and only 2 critical eigenvalues with zero real part and the remaining $m = n - 2$ eigenvalues have negative real parts. Then the system (19) can be transformed to

$$\left\{ \begin{array}{rcl} \dot{u} & = & Bu + g(u,v) \\ \dot{v} & = & Cv + h(u,v) \end{array} \right. \tag{20}$$

where $u \in \Re^2$ and $v \in \Re^m$. B is a 2×2 matrix with its eigenvalues on the imaginary axis and C is a $m \times m$ matrix with no eigenvalues on the imaginary axis. The functions g and h have at least quadratic terms. The centre manifold W^c of system (20) can be locally represented as a graph of a smooth function,

$$W^c = \{(u,v) \quad : v = V(u)\} \tag{21}$$

$V : \Re^2 \to \Re^m$ and due to the tangent property of W^c, $V(u) = O(||u||^2)$.

The Reduction Principle says system (20) is locally topologically equivalent near the origin to

$$\left\{ \begin{array}{rcl} \dot{u} & = & Bu + g(u,V(u)) \\ \dot{v} & = & Cv \end{array} \right. \tag{22}$$

The important thing to notice is that the equations for u and v are decoupled in equation (22). The first equation is the restriction of equation (20) to its centre manifold. The dynamics of the structurally unstable system (20) are essentially determined by this restriction, since the second equation in (22) is linear. For a Hopf bifurcation with ($\lambda_{1,2} = \pm i\omega$) then the system looks like

$$\left\{ \begin{array}{rcl} \begin{pmatrix} \dot{u_1} \\ \dot{u_2} \end{pmatrix} & = & \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} G_1(u_1,u_2,v) \\ G_2(u_1,u_2,v) \end{pmatrix} \\ \dot{v} & = & Cv + H_1(u_1,u_2,v) \end{array} \right. \tag{23}$$

It is possible to rewrite this in complex form by use of the variable $z = u_1 + iu_2$ to obtain

$$\begin{cases} \dot{z} &= i\omega z + G(z, \bar{z}, v) \\ \dot{v} &= Cv + H(z, \bar{z}, v) \end{cases} \tag{24}$$

where G and H are smooth complex-valued functions of z, $\bar{z} \in C^1$ of at least quadratic order. The centre manifold W^c can be locally represented as a graph of a smooth function

$$W^c = \{(z, v) : v = V(z)\}$$

V maps $\Re^2 \to \Re^{n-2}$ and due to the tangent property of W^c, $V(z) = O||u||^2$. The Centre manifold W^c therefore has the representation

$$v = V(z, \bar{z}) = \frac{1}{2} w_{20} z^2 + w_{11} z\bar{z} + \frac{1}{2} w_{02} \bar{z}^2 + O(|z|^3), \tag{25}$$

with the coefficients $w_{ij} \in C^2$. Since v must be real, w_{11} is real and $w_{20} = \bar{w}_{02}$. Using Taylor expansions in z, $\bar{z}$, and v the system (24) can be rewritten as

$$\begin{cases} \dot{z} &= i\omega z + \frac{1}{2} G_{20} z^2 + G_{11} z\bar{z} + \frac{1}{2} G_{02} \bar{z}^2 \\ &+ \frac{1}{2} G_{21} z^2 \bar{z} + \langle G_{10}, v \rangle z + \langle G_{01}, v \rangle \bar{z} + \dots \\ \dot{v} &= Cv + \frac{1}{2} H_{20} z^2 + H_{11} z\bar{z} + \frac{1}{2} H_{02} \bar{z}^2 + \dots \end{cases} \tag{26}$$

where G_{20}, G_{11}, G_{02}, $G_{21} \in C^1$ and G_{01}, G_{10}, $H_{ij} \in C^{n-2}$. Since v is real H_{11} is real, and $H_{20} = \bar{H}_{02}$.

$$G_{jk} = \frac{\partial^{j+k}}{\partial z^j \partial \bar{z}^k} G(z, \bar{z}, 0)\bigg|_{z=0}, \quad j + k \geq 2, \tag{27}$$

$$\bar{G}_{10,j} = \frac{\partial^2}{\partial v_j \partial z} G(z, \bar{z}, v)\bigg|_{z=0, v=0}, \quad j = 1, 2, \dots, n-2, \tag{28}$$

$$\bar{G}_{01,j} = \frac{\partial^2}{\partial v_j \partial \bar{z}} G(z, \bar{z}, v)\bigg|_{z=0, v=0}, \quad j = 1, 2, \dots, n-2, \tag{29}$$

$$H_{jk} = \frac{\partial^{j+k}}{\partial z^j \partial \bar{z}^k} H(z, \bar{z}, 0)\bigg|_{z=0}, \quad j + k = 2, \tag{30}$$

On substituting the representation of the centre manifold in (26) and equating coeffients,

$$\begin{aligned} w_{20} &= (2i\omega I - C)^{-1} H_{20} \\ w_{11} &= -C^{-1} H_{11} \\ w_{02} &= (-2i\omega I - C)^{-1} H_{20} \end{aligned} \tag{31}$$

Where I is the identity matrix and the matrices $(2i\omega I - C)$, C and $(-2i\omega I - C)$ are invertible since 0 and $\pm 2i\omega$ are not eigenvalues of C.

For reduction to be worthwhile the bifurcation parameter must also be added to the system and included in the calculated centre manifolds. This allows the reduced model to be applied for parameter values away from the bifurcation value. Consider the parameterized equation

$$\dot{x} = F(x, \alpha)$$

where $x \in \Re^n$ and $\alpha \in \Re^m$. Suppose that at $\alpha = 0$ the system has a non-hyperbolic equilibrium $x = 0$ which undergoes a Hopf bifurcation. This means we have a system equivalent to

$$\begin{cases} \dot{u} &= Bu + g(u, v, \alpha) \\ \dot{v} &= Cv + h(u, v, \alpha) \end{cases} \tag{32}$$

and since α does not depend on time we can append the equation $\dot{\alpha} = 0$ to the expanded system

$$\begin{cases} \dot{u} &= Bu + g(u, v, \alpha) \\ \dot{v} &= Cv + h(u, v, \alpha) \\ \dot{\alpha} &= 0 \end{cases} \tag{33}$$

The Centre Manifold theorem asserts the existence of a centre manifold for the origin that is local given by points (u, v, α) satisfying an equation of the form

$$v = k(u, \alpha).$$

This is used in the reduction step.