

Matlab code for stochastic model updating

The Matlab code provided performs the stochastic model updating methods described in [1] and [2] using the data obtained from experiments. This document contains a brief description of the theory and the instruction to use the code for the test cases presented.

The perturbation approach

The deterministic finite element model updating problem can be expressed as,

$$\boldsymbol{\theta}_{j+1} = \boldsymbol{\theta}_j + \mathbf{T}_j (\mathbf{z}_m - \mathbf{z}_j) \quad (1)$$

where $\mathbf{z}_j \in \mathbb{R}^{n \times 1}$ is the vector of estimated output parameters (e.g. eigenvalues and eigenvectors), $\mathbf{z}_m \in \mathbb{R}^{n \times 1}$ is the vector of measured data, $\boldsymbol{\theta}$ is the vector of system parameters and \mathbf{T}_j is a transformation matrix. In order to include the variability in measurements, the modal parameters are represented as,

$$\mathbf{z}_m = \bar{\mathbf{z}}_m + \boldsymbol{\delta}\mathbf{z}_m \quad (2)$$

$$\mathbf{z}_j = \bar{\mathbf{z}}_j + \boldsymbol{\delta}\mathbf{z}_j \quad (3)$$

where the overbar denotes mean values and $\boldsymbol{\delta}\mathbf{z}_m, \boldsymbol{\delta}\mathbf{z}_j \in \mathbb{R}^{n \times 1}$ are vectors of random variables. The hyperellipses represented by $\{\bar{\mathbf{z}}_m, \mathbf{cov}(\mathbf{z}_m, \mathbf{z}_m)\}$ and $\{\bar{\mathbf{z}}_j, \mathbf{cov}(\mathbf{z}_j, \mathbf{z}_j)\}$ define the space of measurements and predictions respectively. Correspondingly the variability in physical parameters at the j th iteration is defined as,

$$\boldsymbol{\theta}_j = \bar{\boldsymbol{\theta}}_j + \boldsymbol{\delta}\boldsymbol{\theta}_j \quad (4)$$

and the stochastic model updating problem may then be cast as,

$$\bar{\boldsymbol{\theta}}_{j+1} + \boldsymbol{\delta}\boldsymbol{\theta}_{j+1} = \bar{\boldsymbol{\theta}}_j + \boldsymbol{\delta}\boldsymbol{\theta}_j + (\bar{\mathbf{T}}_j + \boldsymbol{\delta}\mathbf{T}_j) (\bar{\mathbf{z}}_m + \boldsymbol{\delta}\mathbf{z}_m - \bar{\mathbf{z}}_j - \boldsymbol{\delta}\mathbf{z}_j) \quad (5)$$

where the transformation matrix becomes,

$$\mathbf{T}_j = \bar{\mathbf{T}}_j + \boldsymbol{\delta}\mathbf{T}_j = \bar{\mathbf{T}}_j + \sum_{k=1}^n \frac{\partial \bar{\mathbf{T}}_j}{\partial z_{mk}} \delta z_{mk} \quad (6)$$

In the above equations, $\bar{\mathbf{T}}_j$ denotes the transformation matrix at the parameter means, $\bar{\mathbf{T}}_j = \mathbf{T}(\bar{\boldsymbol{\theta}}_j)$, and δz_{mk} denotes the k th element of $\boldsymbol{\delta}\mathbf{z}_m$.

Application of the perturbation method, by separating the zeroth order and first order terms from eq. (5) leads to,

$$\mathbf{O}(\delta^0) : \quad \bar{\boldsymbol{\theta}}_{j+1} = \bar{\boldsymbol{\theta}}_j + \bar{\mathbf{T}}_j (\bar{\mathbf{z}}_m - \bar{\mathbf{z}}_j) \quad (7)$$

$$\mathbf{O}(\delta^1) : \quad \boldsymbol{\delta}\boldsymbol{\theta}_{j+1} = \boldsymbol{\delta}\boldsymbol{\theta}_j + \bar{\mathbf{T}}_j (\boldsymbol{\delta}\mathbf{z}_m - \boldsymbol{\delta}\mathbf{z}_j) + \left(\sum_{k=1}^n \frac{\partial \bar{\mathbf{T}}_j}{\partial z_{mk}} \delta z_{mk} \right) (\bar{\mathbf{z}}_m - \bar{\mathbf{z}}_j) \quad (8)$$

Eq. (7) leads to the estimate of the mean of the parameters and eq. (8) is used in the determination of the covariance matrix. Two version of perturbation methods have developed for estimation of covariance matrix. The details can be found in [1]. The simplified version of perturbation method that is used in the plate example consists of two recursive equations having the following form for the estimation of the mean values and co-variances of the parameters are obtained as,

$$\bar{\boldsymbol{\theta}}_{j+1} = \bar{\boldsymbol{\theta}}_j + \bar{\mathbf{T}}_j (\bar{\mathbf{z}}_m - \bar{\mathbf{z}}_j) \quad (9)$$

$$\begin{aligned} \mathbf{cov}(\boldsymbol{\delta}\boldsymbol{\theta}_{j+1}, \boldsymbol{\delta}\boldsymbol{\theta}_{j+1}) &= \mathbf{cov}(\boldsymbol{\delta}\boldsymbol{\theta}_j, \boldsymbol{\delta}\boldsymbol{\theta}_j) - \mathbf{cov}(\boldsymbol{\delta}\boldsymbol{\theta}_j, \boldsymbol{\delta}\mathbf{z}_j) \bar{\mathbf{T}}_j^T + \bar{\mathbf{T}}_j \mathbf{cov}(\boldsymbol{\delta}\mathbf{z}_m, \boldsymbol{\delta}\mathbf{z}_m) \bar{\mathbf{T}}_j^T \\ &- \bar{\mathbf{T}}_j \mathbf{cov}(\boldsymbol{\delta}\mathbf{z}_j, \boldsymbol{\delta}\boldsymbol{\theta}_j) + \bar{\mathbf{T}}_j \mathbf{cov}(\boldsymbol{\delta}\mathbf{z}_j, \boldsymbol{\delta}\mathbf{z}_j) \bar{\mathbf{T}}_j^T \end{aligned} \quad (10)$$

In the case of ill-conditioned model-updating equations, the transformation matrix may be expressed as the weighted pseudo inverse as,

$$\bar{\mathbf{T}}_j = (\bar{\mathbf{S}}_j^T \mathbf{W}_1 \bar{\mathbf{S}}_j + \mathbf{W}_2)^{-1} \bar{\mathbf{S}}_j^T \mathbf{W}_1 \quad (11)$$

the minimum-norm regularised solution is obtained when $\mathbf{W}_2 = \lambda \mathbf{I}$ and λ is the regularisation parameter that locates the corner of the L-curve obtained by plotting the norms $\|\boldsymbol{\theta}_{j+1} - \boldsymbol{\theta}_j\|$ vs. $\|\bar{\mathbf{S}}_j (\boldsymbol{\theta}_{j+1} - \boldsymbol{\theta}_j) - (\bar{\mathbf{z}}_m - \bar{\mathbf{z}}_j)\|$.

Minimisation of an objective function

As mentioned earlier, the hyperellipses represented by $\{\bar{\mathbf{z}}_m, \mathbf{cov}(\mathbf{z}_m, \mathbf{z}_m)\}$ and $\{\bar{\mathbf{z}}_j, \mathbf{cov}(\mathbf{z}_j, \mathbf{z}_j)\}$ define the space of measurements and predictions respectively. In order to minimise the distance and also the size difference in between these two spaces, we propose an objective function as,

$$F = (\bar{\mathbf{z}}_m - \bar{\mathbf{z}}_j)^T \mathbf{W}_1 (\bar{\mathbf{z}}_m - \bar{\mathbf{z}}_j) + w_2 \|\mathbf{cov}(\mathbf{z}_m, \mathbf{z}_m) - \mathbf{cov}(\mathbf{z}_j, \mathbf{z}_j)\|_F \quad (12)$$

where $\|\bullet\|_F$ is Frobenius norm, $\bar{\mathbf{z}}_m$ is estimated mean values of test results, $\mathbf{cov}(\mathbf{z}_m, \mathbf{z}_m)$ is the covariance matrix of measured data, $\bar{\mathbf{z}}_m$ and $\mathbf{cov}(\mathbf{z}_m, \mathbf{z}_m)$ are the estimated mean values and the covariance matrix of predictions from mathematical model at j th iteration respectively. Therefore the stochastic model updating problem can be expressed as,

$$\min_{\bar{\boldsymbol{\theta}}, \boldsymbol{\sigma}_\theta} \left[(\bar{\mathbf{z}}_m - \bar{\mathbf{z}}_j)^T \mathbf{W}_1 (\bar{\mathbf{z}}_m - \bar{\mathbf{z}}_j) + w_2 \|\mathbf{cov}(\mathbf{z}_m, \mathbf{z}_m) - \mathbf{cov}(\mathbf{z}_j, \mathbf{z}_j)\|_F \right] \quad (13)$$

subject to :

$$\bar{\boldsymbol{\theta}} > 0 \text{ and } \boldsymbol{\sigma}_\theta > 0$$

where $\bar{\boldsymbol{\theta}}$ denotes the mean values and $\boldsymbol{\sigma}_\theta$ is the standard deviations of the system parameters. The weighting matrix, \mathbf{W}_1 , and weighting coefficient, w_2 , may be chosen to make two terms in objective function as the same order.

Experimental case studies

Case1: Aluminum plates with random thicknesses

Ten aluminum plates were prepared so that a contrived distribution of thicknesses, close to Gaussian, was obtained by machining. Care was taken to try to obtain a constant thickness

for each plate but was not achieved perfectly. The mean value of the thicknesses was 3.975mm with a standard deviation of 0.163mm. In the experimental set up free boundary conditions were used to avoid the introduction of other uncertainties due to clamping or pinning at the edges of the plates. All ten plates had the same overall dimensions, length 0.4m and width 0.1m. A hammer test was carried out using four uniaxial fixed accelerometers. Figure 1 shows the excitation point, marked F, and the positions of four accelerometers, marked A, B, C and D. The mass of each accelerometer was 2 grams represented by lumped masses in the finite element model. The first 10 measured natural frequencies of all ten plates are given in Tables 1 and 2.

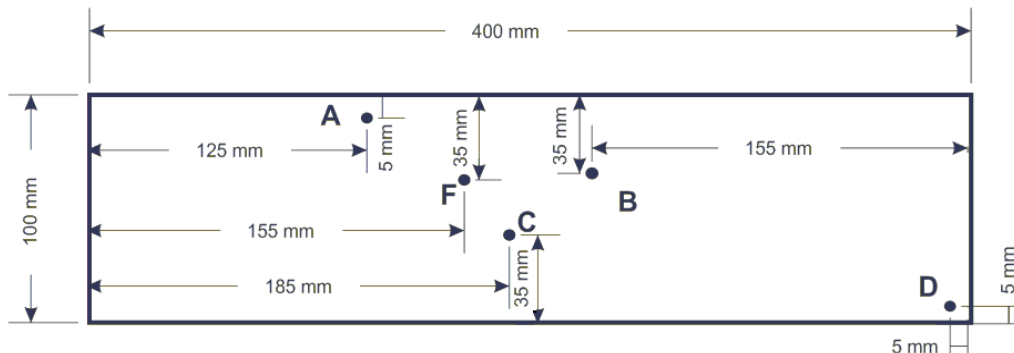


Figure 1: Arrangement of accelerometers (A, B, C, D) and excitation point (F)

Table 1: The first five measured natural frequencies (Hz) for the ten plates

| Plate number | Mode Number | | | | |
|--------------|-------------|---------|---------|---------|---------|
| | 1 | 2 | 3 | 4 | 5 |
| 1 | 119.774 | 284.283 | 331.970 | 589.404 | 656.359 |
| 2 | 121.615 | 291.922 | 337.186 | 605.160 | 665.854 |
| 3 | 123.156 | 291.440 | 340.184 | 602.603 | 673.357 |
| 4 | 128.048 | 298.163 | 355.210 | 620.139 | 700.798 |
| 5 | 128.533 | 303.809 | 357.110 | 630.809 | 704.505 |
| 6 | 128.596 | 301.010 | 361.488 | 635.533 | 713.207 |
| 7 | 129.796 | 311.726 | 361.114 | 646.765 | 712.792 |
| 8 | 135.058 | 315.393 | 374.368 | 653.584 | 738.395 |
| 9 | 134.478 | 312.215 | 374.406 | 649.130 | 737.256 |
| 10 | 138.141 | 321.812 | 382.932 | 667.203 | 755.189 |
| Mean | 128.720 | 303.177 | 357.597 | 630.033 | 705.771 |
| Std | 6.011 | 12.032 | 17.048 | 25.235 | 32.854 |

The thickness variations were measured using a long-jaw micrometer at 4×14 points and can be found in *measthickplate.zip* file.

Table 2: The 6th to 10th measured natural frequencies (Hz) for the ten plates

| Mode Number | | | | | |
|--------------|----------|----------|----------|----------|----------|
| Plate number | 6 | 7 | 8 | 9 | 10 |
| 1 | 932.576 | 1091.603 | 1343.097 | 1628.879 | 1825.215 |
| 2 | 953.666 | 1106.861 | 1372.890 | 1650.395 | 1860.225 |
| 3 | 955.515 | 1119.445 | 1376.298 | 1669.899 | 1868.071 |
| 4 | 980.403 | 1165.177 | 1414.181 | 1736.714 | 1924.260 |
| 5 | 995.188 | 1169.660 | 1433.020 | 1743.750 | 1946.155 |
| 6 | 999.248 | 1184.455 | 1440.134 | 1765.415 | 1957.581 |
| 7 | 1019.052 | 1184.608 | 1467.366 | 1766.361 | 1987.556 |
| 8 | 1031.837 | 1225.375 | 1487.512 | 1825.602 | 2021.640 |
| 9 | 1023.229 | 1224.420 | 1479.268 | 1824.121 | 2013.354 |
| 10 | 1053.974 | 1253.610 | 1519.011 | 1866.665 | 2031.377 |
| Mean | 994.469 | 1172.521 | 1433.278 | 1747.780 | 1943.543 |
| Std | 38.877 | 53.840 | 56.771 | 79.232 | 72.908 |

Case2: Aluminum plates with random masses

Thirteen sets of masses having a distribution close to Gaussian were prepared. The experimental set up is shown in Figure 2. Each set included eight equal masses. The 11.5 gram set, for example, included eight masses all of 11.5 grams. The distribution of nominal masses is shown in Figure 3. The mean value of the masses was 10.063 grams with a standard deviation of 2.798 grams. Each set was glued to the surface of a plate and a hammer test was carried. The experimental set up and the positions of accelerometers and excitation points were the same as previous case study. The positions of added masses on the plate are shown in Figure 4. Each of added mass and mass of the accelerometer were represented by lumped masses in the finite element model. The first six natural frequencies of all 13 sets are given in Table 3. The stochastic model updating by minimising an objective function can be used in this example.

Using the code

stochasticupdating.m is the MATLAB code for optimisation procedure of both cases that allow the user to choose one of the aforementioned cases. In case 1, user can provide some data (e.g. number of parameters, number of samples and etc) or choose the default values for the case. The code will then produce graphs given in [1] (figure29 in [1]). If case 2 is selected optimisation toolbox of MATLAB including the problem of plate with 13 samples (plates with random masses) will be opened. The user need to click on the start button to execute the program.

It should be noted that for each case following MATLAB files should be used when the code

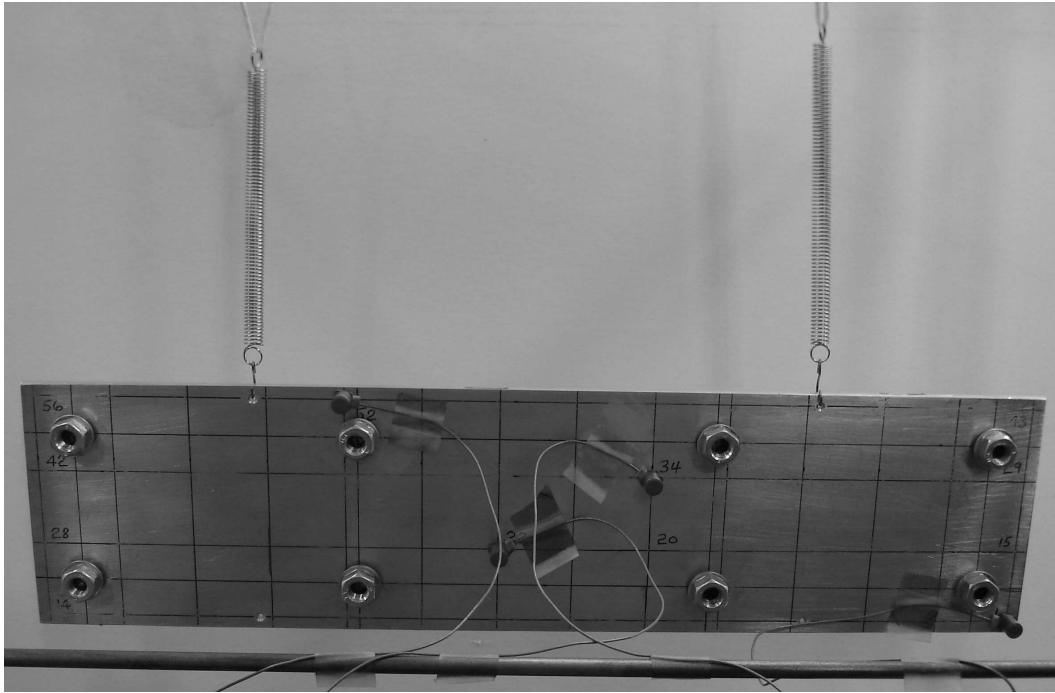


Figure 2: Experimental setup

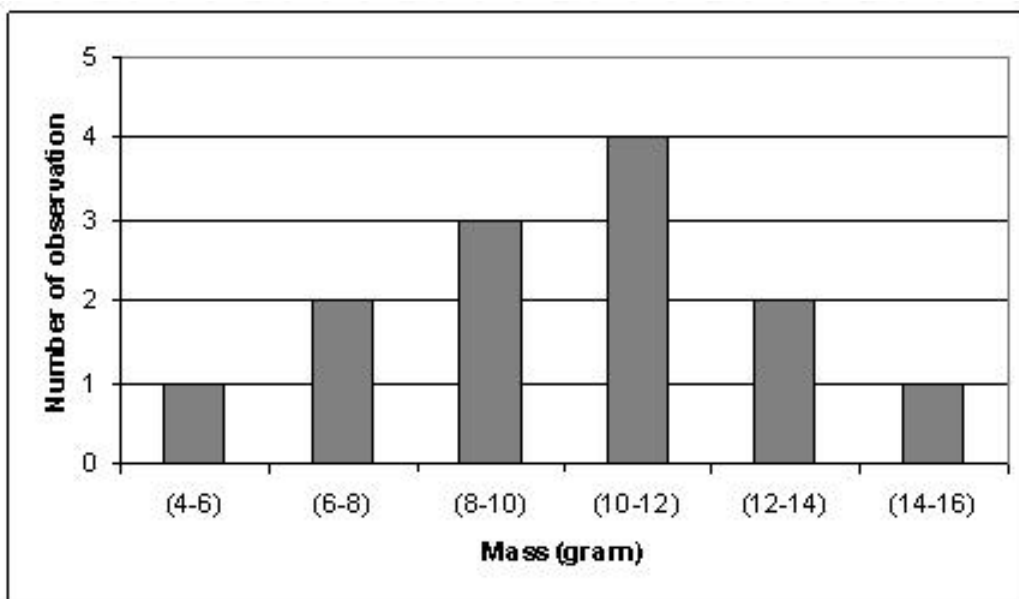


Figure 3: Distribution of masses

is run,

case1.zip consists of:

fem.m: finite element model of plate. This file produces eigenvectors and eigenvalues from modal analysis of FE model. These data can be also provided from commercial software, e.g. sol 103 in MSC NASTRAN.

sens.m: produces sensitivity matrices with respect to updating parameters (thicknesses in

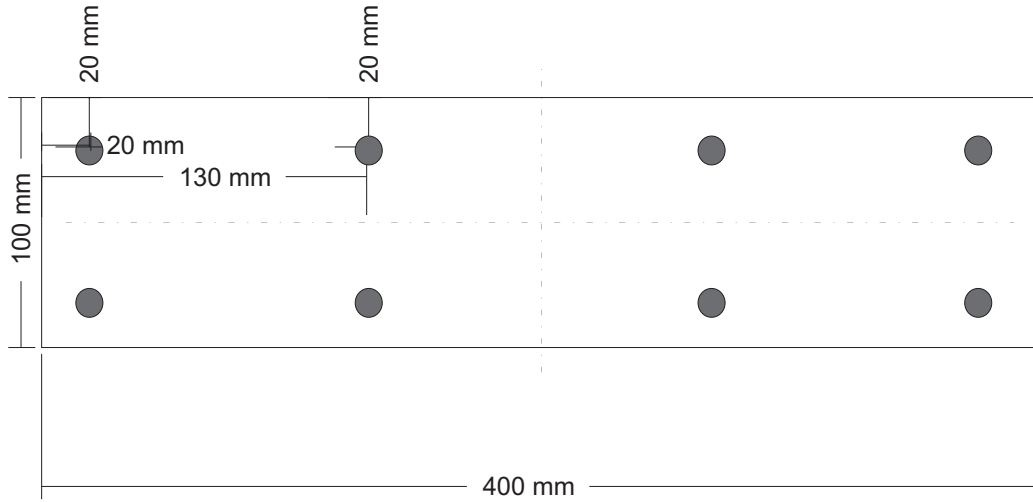


Figure 4: The positions of the masses on the plate

Table 3: The first six measured natural frequencies (Hz) for a plate with 13 different sets of 8 masses attached.

| Mass (grams) | Mode Number | | | | | |
|--------------|-------------|---------|---------|---------|---------|---------|
| | 1 | 2 | 3 | 4 | 5 | 6 |
| 5.025 | 121.080 | 286.799 | 333.896 | 595.693 | 688.093 | 915.365 |
| 6.588 | 119.002 | 280.460 | 327.573 | 585.042 | 684.618 | 894.911 |
| 7.538 | 117.817 | 277.315 | 323.931 | 579.240 | 681.073 | 882.836 |
| 8.55 | 116.385 | 272.994 | 319.427 | 570.238 | 674.886 | 864.382 |
| 9.088 | 115.659 | 271.367 | 317.253 | 566.972 | 672.319 | 858.409 |
| 9.563 | 115.071 | 270.059 | 315.601 | 564.025 | 670.297 | 851.946 |
| 10.075 | 114.413 | 267.771 | 313.152 | 558.999 | 663.869 | 844.604 |
| 10.613 | 113.766 | 266.462 | 311.447 | 555.173 | 660.905 | 833.890 |
| 11.113 | 113.021 | 264.995 | 309.576 | 552.080 | 662.606 | 828.573 |
| 11.5 | 112.802 | 264.543 | 308.426 | 552.121 | 662.895 | 836.105 |
| 12.575 | 111.514 | 261.684 | 304.884 | 544.291 | 655.675 | 813.238 |
| 13.575 | 110.809 | 259.442 | 302.668 | 541.900 | 660.888 | 808.048 |
| 15.013 | 108.870 | 254.557 | 296.379 | 528.127 | 639.655 | 777.946 |
| Mean | 114.632 | 269.111 | 314.170 | 561.069 | 667.522 | 846.943 |
| Std | 3.409 | 8.837 | 10.412 | 18.631 | 13.063 | 37.385 |

this case). The Sensitivity matrix can be also obtained from Sol 200 MSC NASTRAN. *measured-case1.mat*: includes complete set of measured frequencies from 10 samples.

case2.zip consists of:

fem.m: finite element model of plate with masses. This file produces eigenvectors and eigenvalues from modal analysis of FE model. These data can also be provided from commercial software, e.g. sol 103 in MSC NASTRAN.

sens.m: produces sensitivity matrices with respect to updating parameters (masses in this case). The Sensitivity matrix can also be obtained from Sol 200 MSC NASTRAN.

measured-case2.mat: includes complete set of measured frequencies from 13 samples.

options.mat : the problem and options for *optimtool*

References

- [1] H.H. Khodaparast, J.E. Mottershead, and M.I. Friswell. Perturbation methods for the estimation of parameter variability in stochastic model updating. *Mechanical Systems and Signal Processing*, 22(8):1751–1773, 2008.
- [2] H.H. Khodaparast and J.E. Mottershead. Efficient methods in stochastic model updating. In *Proceedings of International Conference on Noise and Vibration, ISMA2008*, Leuven ,Belgium, 2008.