Taylor Series Expansion- and Least Square- Based Lattice Boltzmann Method

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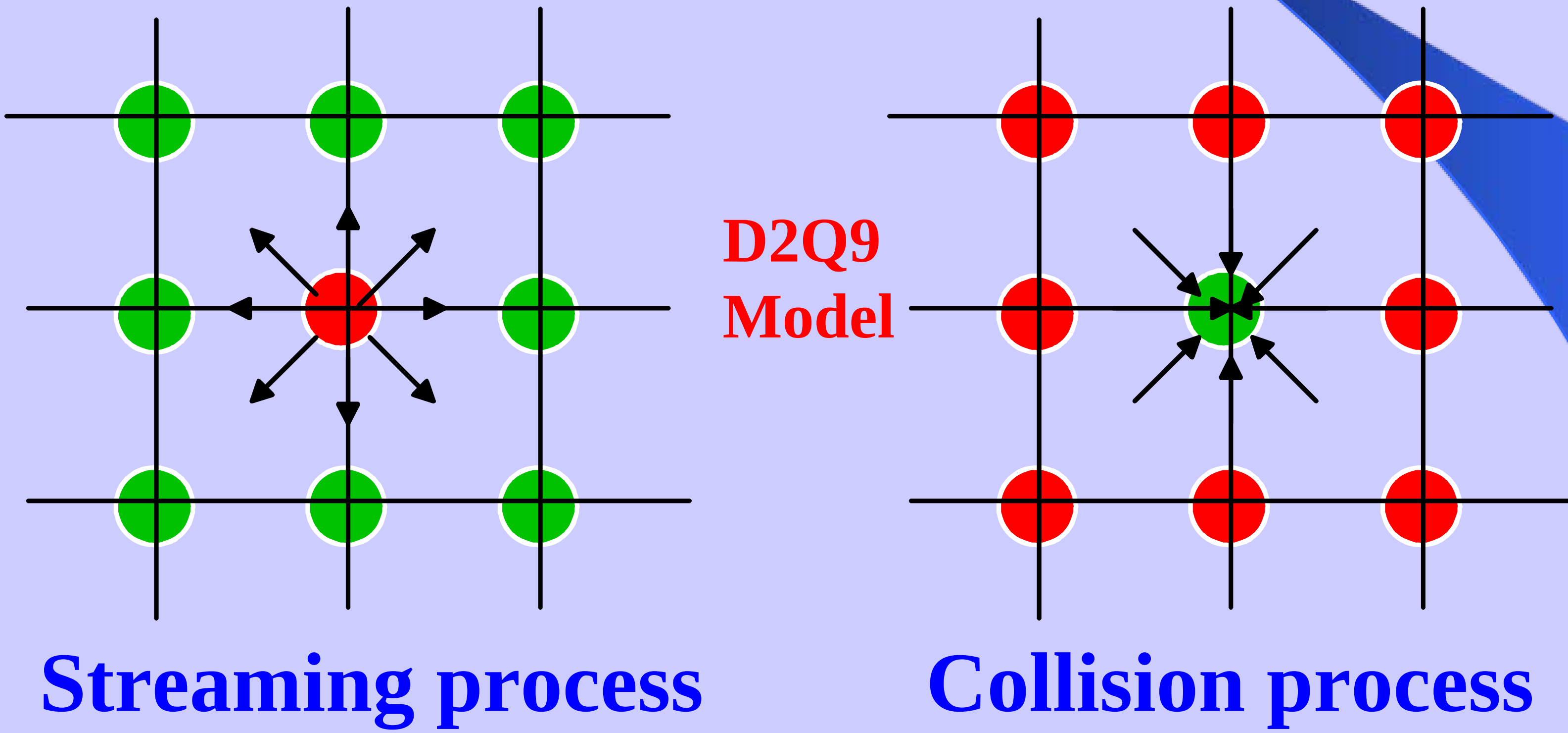
- Standard Lattice Boltzmann Method (LBM)
- Current LBM Methods for Complex Problems
- Taylor Series Expansion- and Least Square-Based LBM (TLLBM)
- Some Numerical Examples
- Conclusions

1. Standard Lattice Boltzmann Method (LBM)

- Particle-based Method (streaming & collision)



$$f_\alpha(x + e_{\alpha x}\delta t, y + e_{\alpha y}\delta t, t + \delta t) = f_\alpha(x, y, t) + [f_\alpha^{eq}(x, y, t) - f_\alpha(x, y, t)]/\tau$$

$$f_\alpha^{eq} = \rho\left[\frac{1}{2} + \frac{1}{6}\left(2\frac{\mathbf{e}_\alpha \cdot \mathbf{U}}{c^2} + 4\left(\frac{\mathbf{e}_\alpha \cdot \mathbf{U}}{c^2}\right)^2 - \frac{\mathbf{U}^2}{c^2}\right)\right]$$

$$\rho = \sum_{\alpha=0}^{N} f_\alpha$$

$$\rho\mathbf{U} = \sum_{\alpha=0}^{N} f_\alpha \mathbf{e}_\alpha$$

$$P = \rho c^2 / 2$$

$$\upsilon = \frac{(2\tau - 1)}{8} c^2 \delta t$$

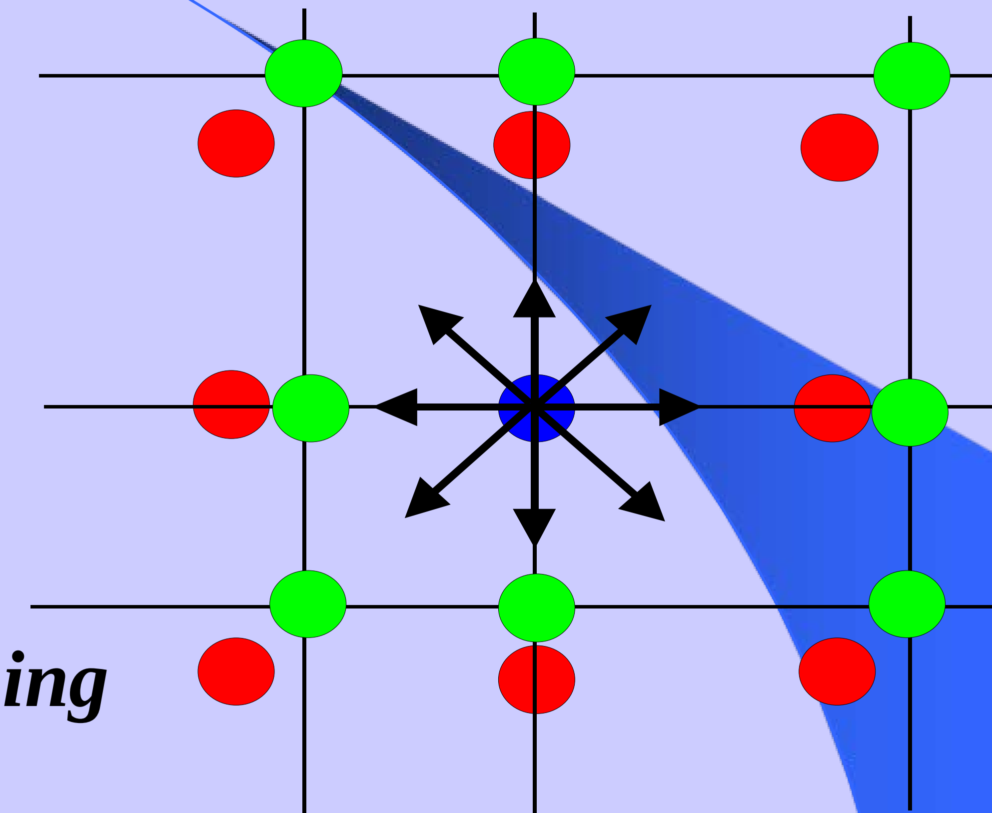
- Features of Standard LBM

  - Particle-based method
  - Only one dependent variable

    Density distribution function f(x,y,t)
  - Explicit updating; Algebraic operation; Easy implementation

    No solution of differential equations and resultant algebraic equations is involved
  - Natural for parallel computing

- Limitation----
  Difficult for complex geometry and non-uniform mesh



- *Mesh points* (green)
- *Positions from streaming* (red)

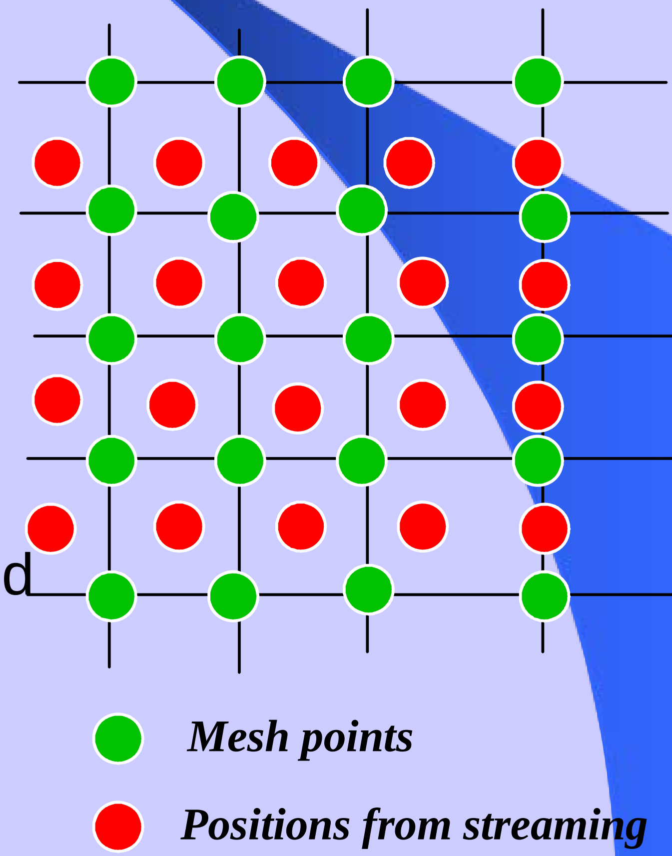
2. Current LBM Methods for Complex Problems

- Interpolation-Supplemented LBM (ISLBM)

He et al. (1996), JCP

Features of ISLBM

- Large computational effort
- May not satisfy conservation Laws at mesh points
- Upwind interpolation is needed for stability



- Differential LBM

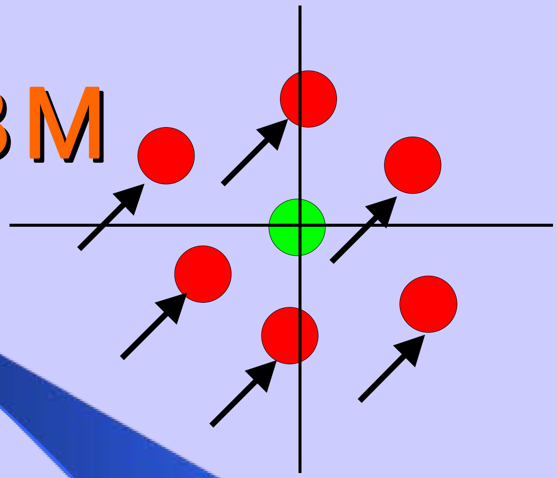
Taylor series expansion to 1st order derivatives

$$\frac{\partial f_\alpha}{\partial t} + e_{\alpha x}\frac{\partial f_\alpha}{\partial x} + e_{\alpha y}\frac{\partial f_\alpha}{\partial y} = \frac{f_\alpha^{eq}(x,y,t) - f_\alpha(x,y,t)}{\tau \cdot \delta t}$$

Features:

- Wave-like equation
- Solved by FD, FE and FV methods
- Artificial viscosity is too large at high Re
- Lose primary advantage of standard LBM (solve PDE and resultant algebraic equations)

3. Development of TLLBM



- Taylor series expansion

$$f_\alpha(A, t+\delta t) = f_\alpha(P, t+\delta t) + \Delta x_A \frac{\partial f_\alpha(P, t+\delta t)}{\partial x} + \Delta y_A \frac{\partial f_\alpha(P, t+\delta t)}{\partial y} +$$

$$\frac{1}{2}(\Delta x_A)^2 \frac{\partial^2 f_\alpha(P, t+\delta t)}{\partial x^2} + \frac{1}{2}(\Delta y_A)^2 \frac{\partial^2 f_\alpha(P, t+\delta t)}{\partial y^2} +$$

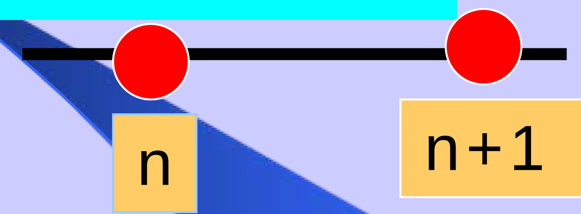
$$\Delta x_A \Delta y_A \frac{\partial^2 f_\alpha(P, t+\delta t)}{\partial x \partial y} + O[(\Delta x_A)^3, (\Delta y_A)^3]$$

P-----Green (objective point)
***A*----Red (neighboring point)**

Drawback: Evaluation of Derivatives

Idea of Runge-Kutta Method (RKM)

$$\frac{du}{dt} = f(u,t), \qquad u = u_0, \quad when \ t = 0$$

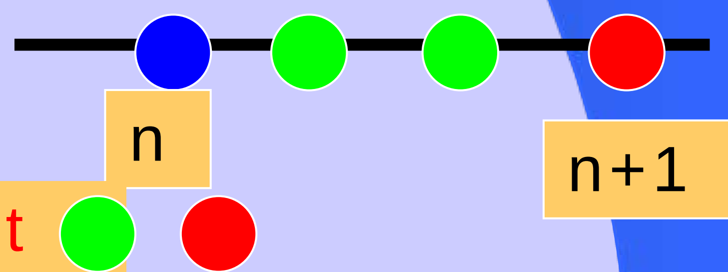


Taylor series method:

$$u^{n+1} = u^n + h \cdot \frac{du}{dt} + \frac{1}{2} h^2 \frac{d^2u}{dt^2} + \frac{1}{6} h^3 \frac{d^3u}{dt^3} + \cdots, \quad h = \Delta t$$

Need to evaluate high order derivatives

Runge-Kutta method:



Apply Taylor series expansion at
Points to form an equation system

- Taylor series expansion is applied at 6 neighbouring points to form an algebraic equation system

A matrix formulation obtained:

$$[S]\{V\} = \{g\} \qquad (*)$$

$$\{V\} = \{f_\alpha, \partial f_\alpha/\partial x, \partial f_\alpha/\partial y, \partial^2 f_\alpha/\partial x^2, \partial^2 f_\alpha/\partial^2 y, \partial^2 f_\alpha/\partial x \partial y\}^T$$

$$\{g\} = \{g_i\}^T \quad g_i = f_\alpha(x_i, y_i, t) + \left[f_\alpha^{eq}(x_i, y_i, t) - f_\alpha(x_i, y_i, t)\right]/\tau$$

[S] is a 6x6 matrix and only depends on the geometric coordinates (calculated in advance in programming)

Least Square Optimization

Equation system (*) may be ill-conditioned or singular (e.g. Points coincide)

Square sum of errors

$$\boldsymbol{E} = \sum_{\boldsymbol{i}=0}^{\boldsymbol{M}} \boldsymbol{err}_{\boldsymbol{i}}^{2} = \sum_{\boldsymbol{i}=0}^{\boldsymbol{M}} \left(\boldsymbol{g}_{\boldsymbol{i}} - \sum_{\boldsymbol{j}=1}^{6} \boldsymbol{s}_{\boldsymbol{i},\boldsymbol{j}} \boldsymbol{V}_{\boldsymbol{j}} \right)^{2}$$

$$i = 0,1,2,\ldots,M \, (M > 5 \ for \ 2D)$$

M is the number of neighbouring points used

Minimize error:

$$\partial \boldsymbol{E} / \partial \boldsymbol{V}_{\boldsymbol{k}} = 0, \boldsymbol{k} = 1,2,\ldots,6$$

Least Square Method (continue)

The final matrix form:

$$\{V\} = \left([S]^T[S]\right)^{-1}[S]^T\{g\} = [A]\{g\}$$

[*A*] is a 6×(*M*+1) matrix

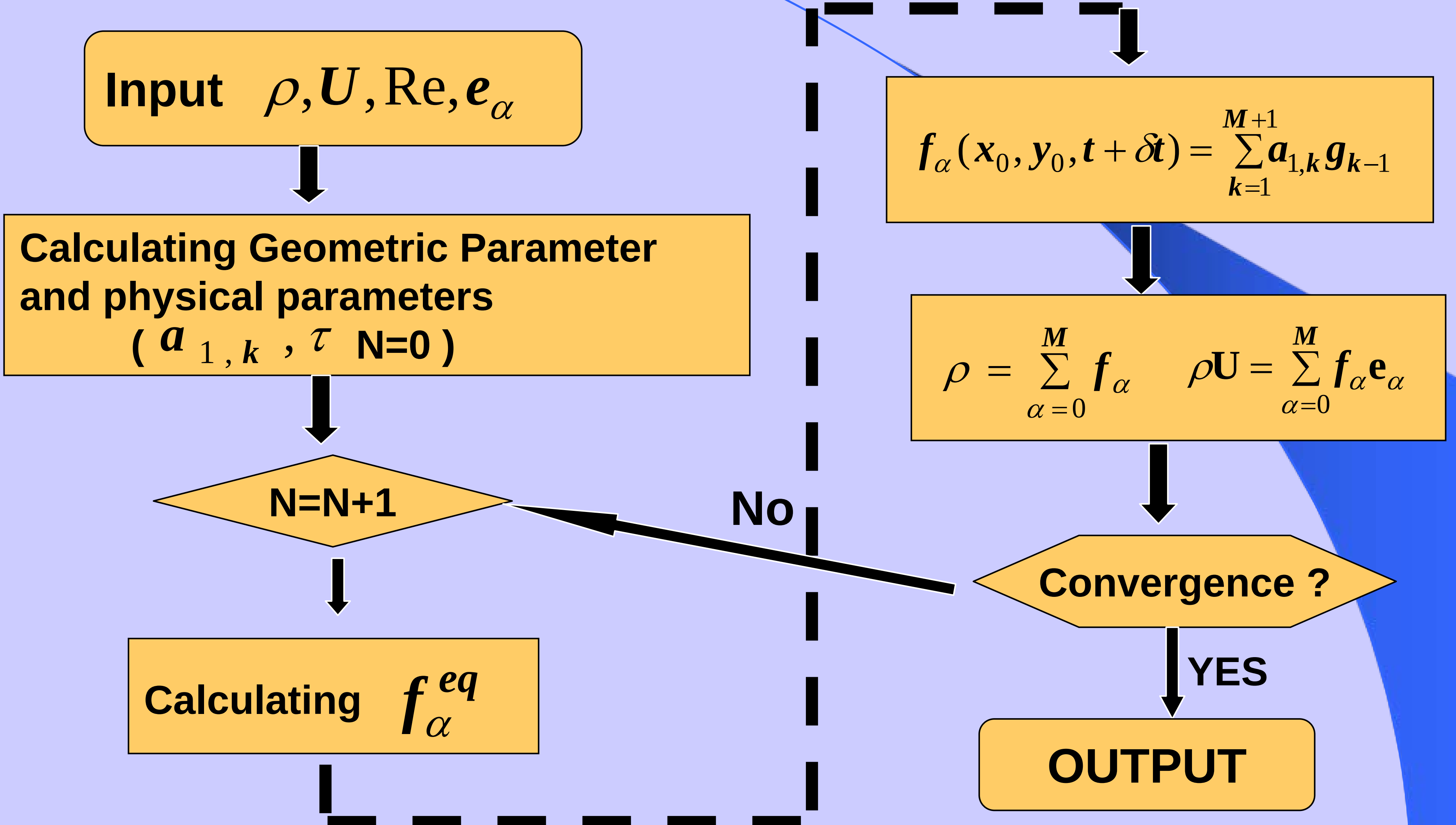
The final explicit algebraic form:

$$f_\alpha(x_0, y_0, t + \delta t) = \sum_{k=1}^{M+1} a_{1,k}\, g_{k-1}$$

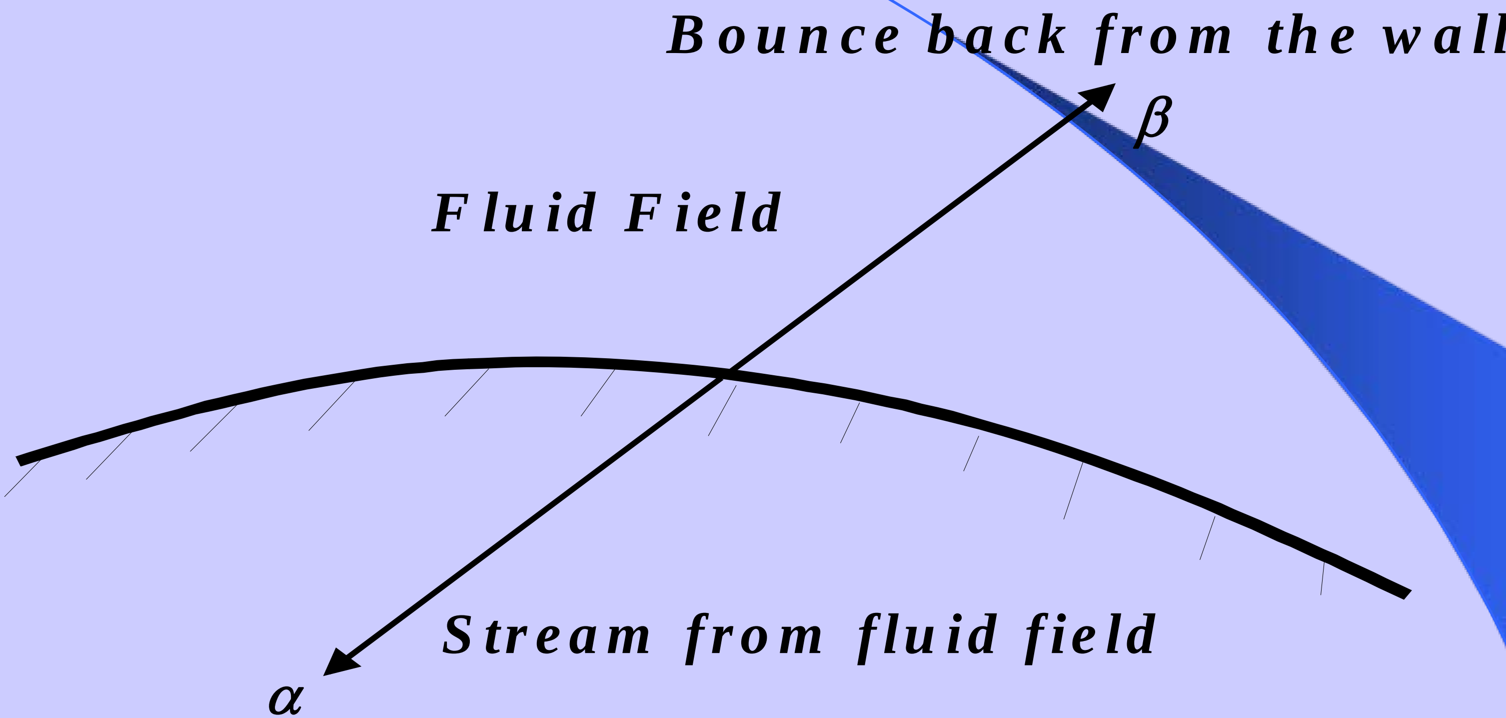
$a_{1,k}$ are the elements of the first row of the matrix [*A*] (pre-computed in program)

- Features of TLLBM
  - Keep all advantages of standard LBM
  - Mesh-free
  - Applicable to any complex geometry
  - Easy application to different lattice models

Flow Chart of Computation



Boundary Treatment

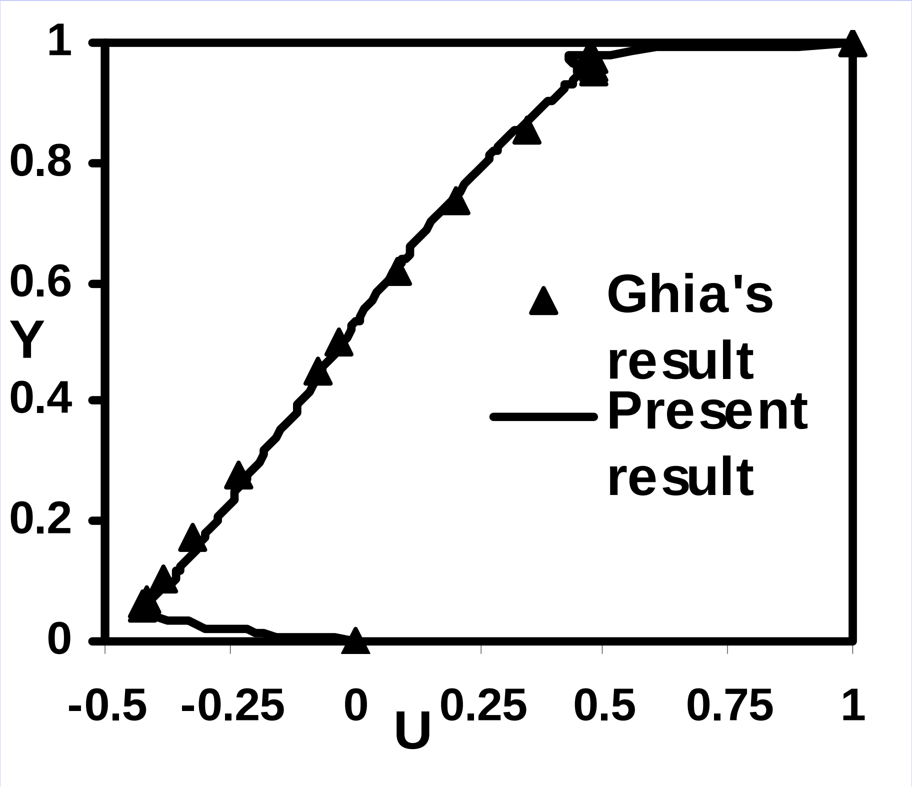


$$f_\beta = f_\alpha$$

Non-slip condition is exactly satisfied

4. Some Numerical Examples

Square Driven Cavity (Re=10,000, Non-uniform mesh 145x145)



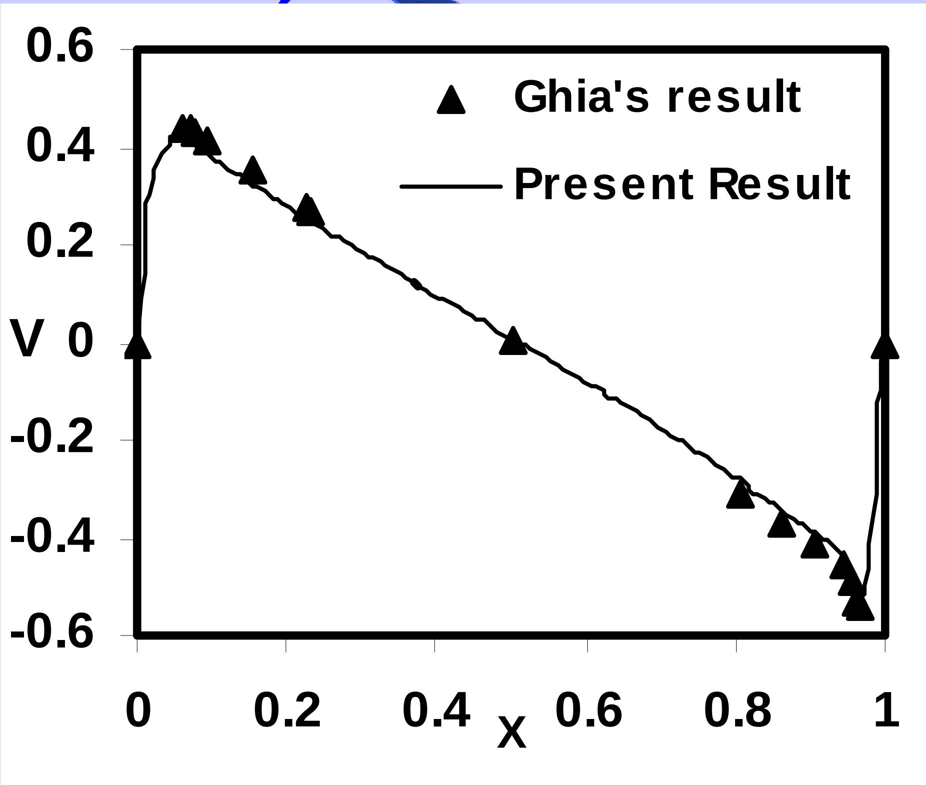


Fig.1 velocity profiles along vertical and horizontal central lines

Square Driven Cavity (Re=10,000, Non-uniform mesh 145x145)

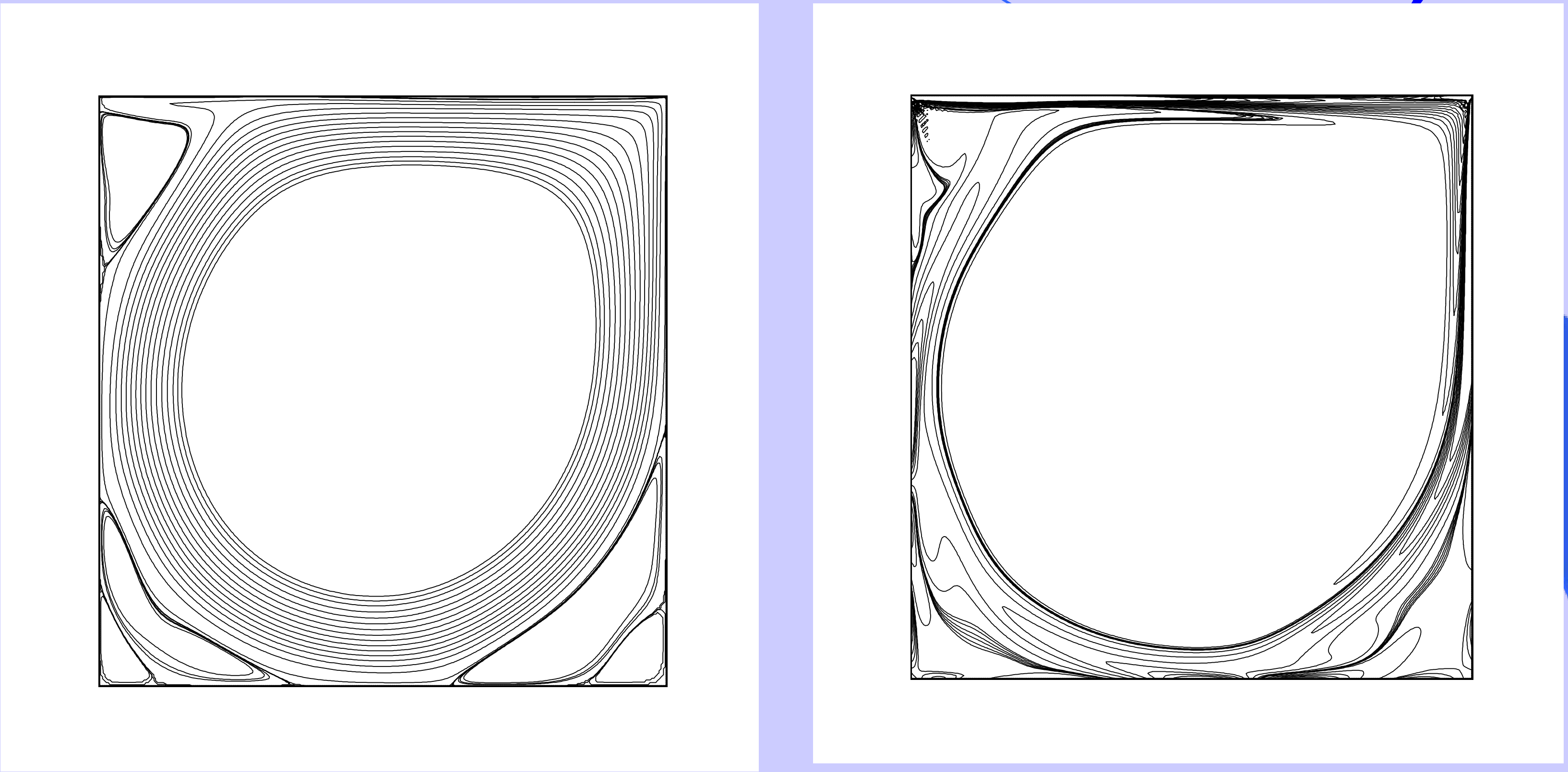


Fig.2 Streamlines (right) and Vorticity contour (left)

Lid-Driven Polar Cavity Flow

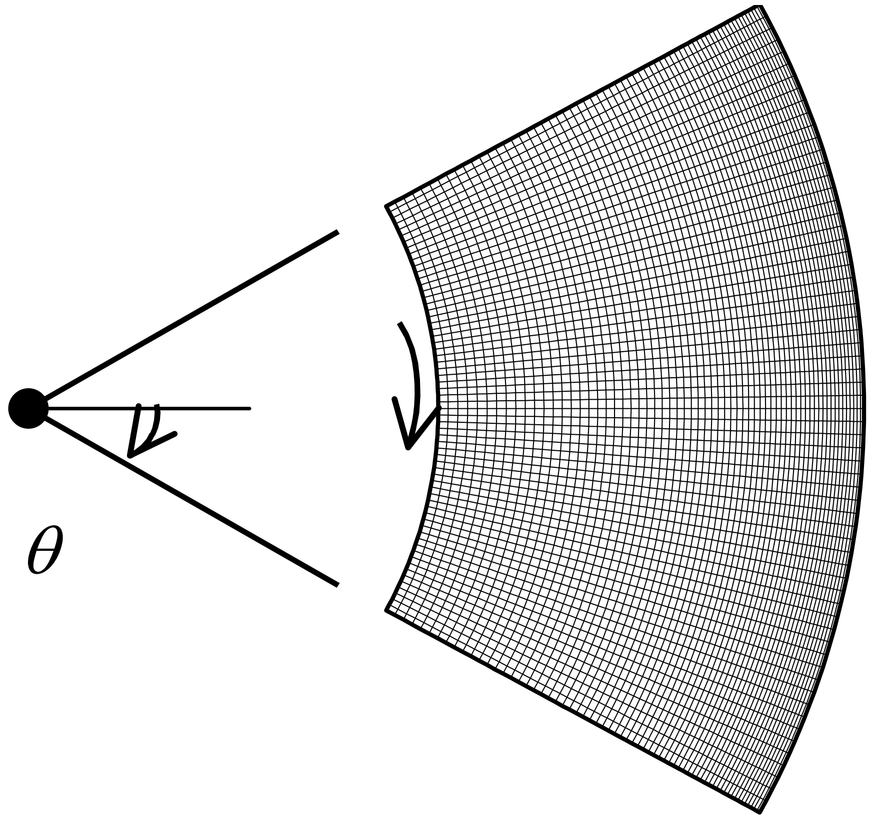


Fig. 3 Sketch of polar cavity and mesh

Lid-Driven Polar Cavity Flow

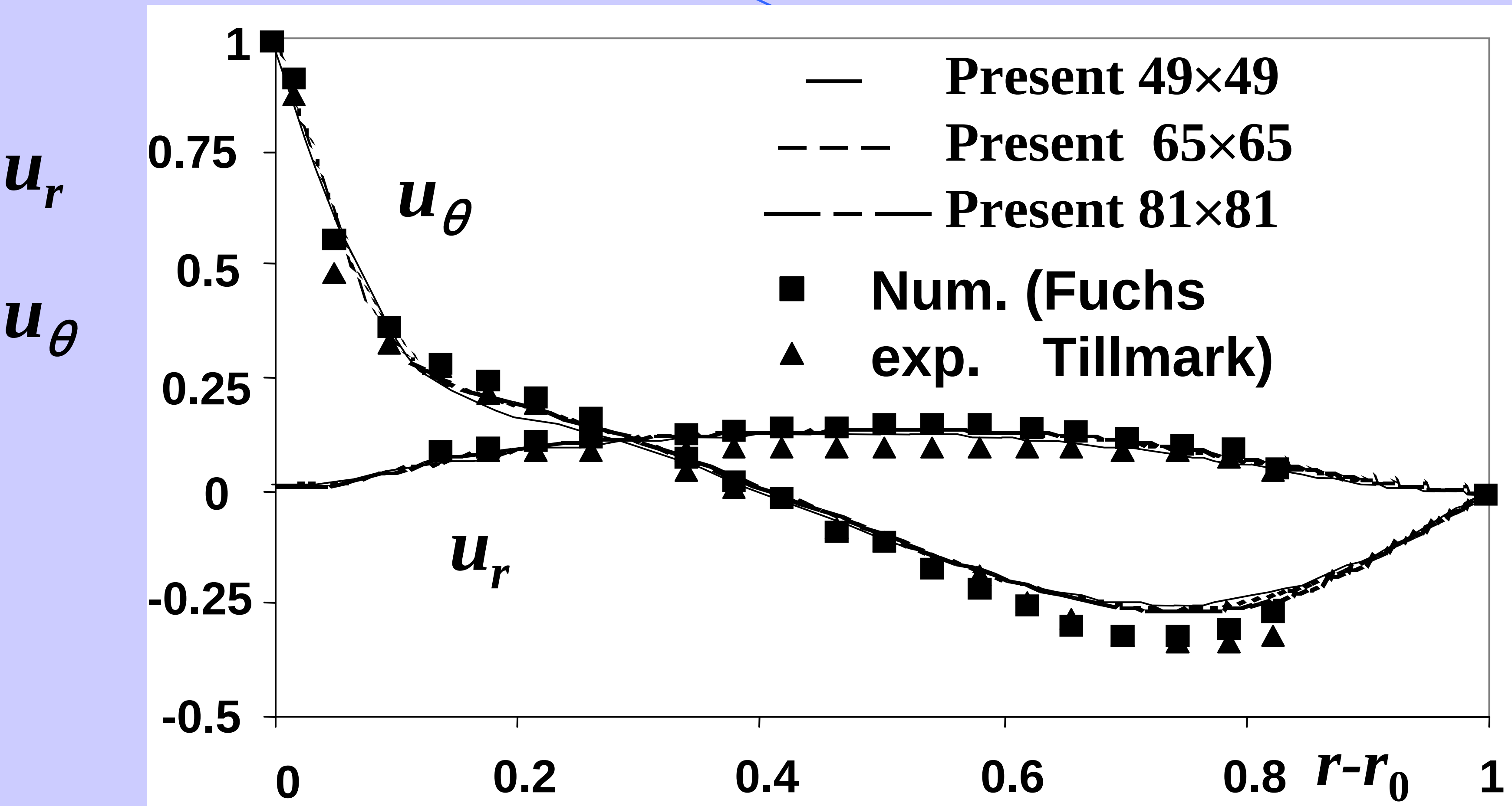


Fig.4 Radial and azimuthal velocity profile along the line of θ=0^0 with Re=350

Lid-Driven Polar Cavity Flow

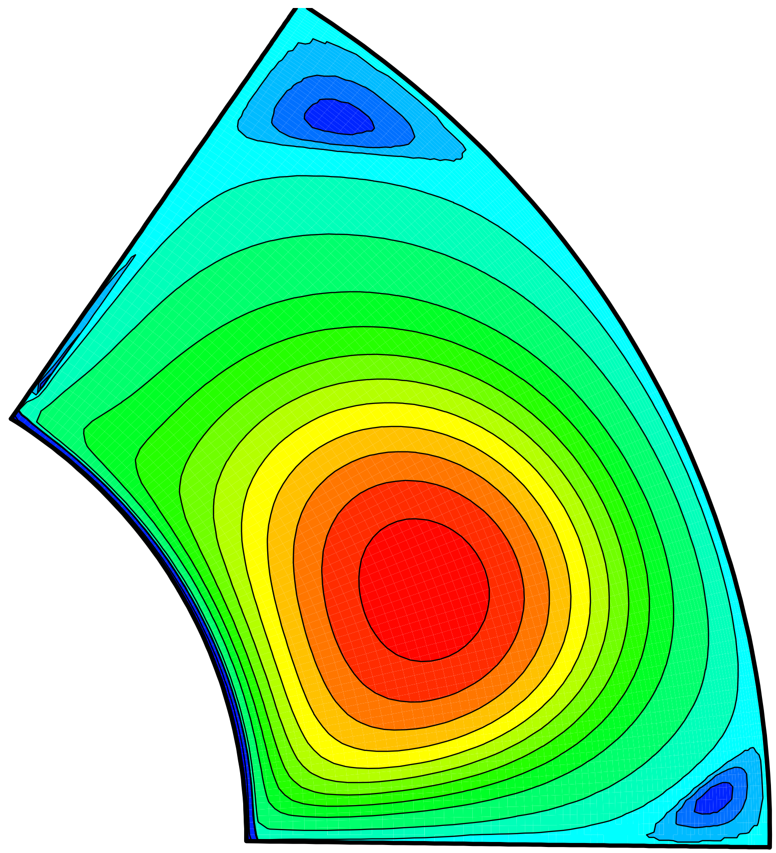


Fig. 5 Streamlines in Polar Cavity for Re=350

Flow around A Circular Cylinder

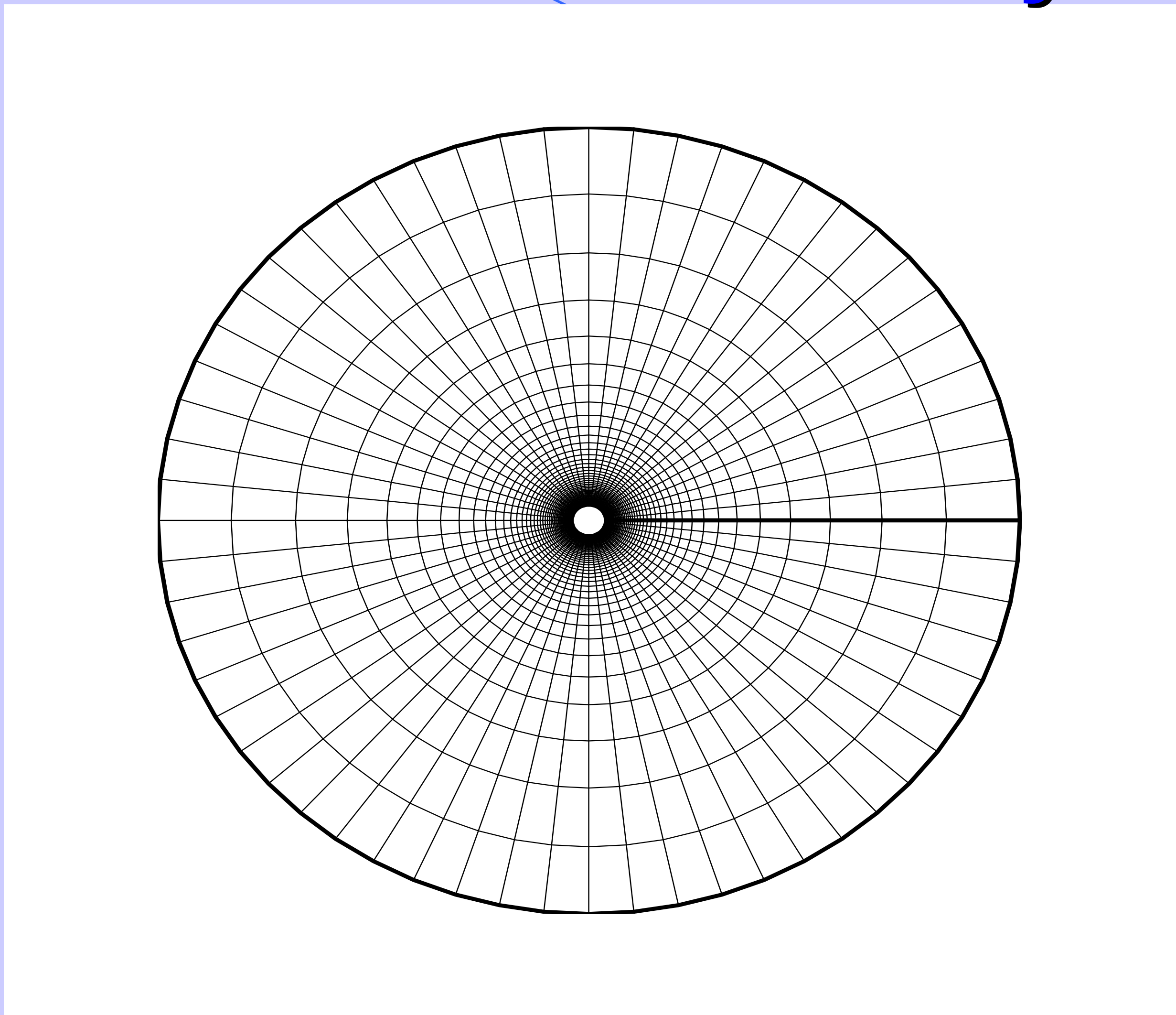
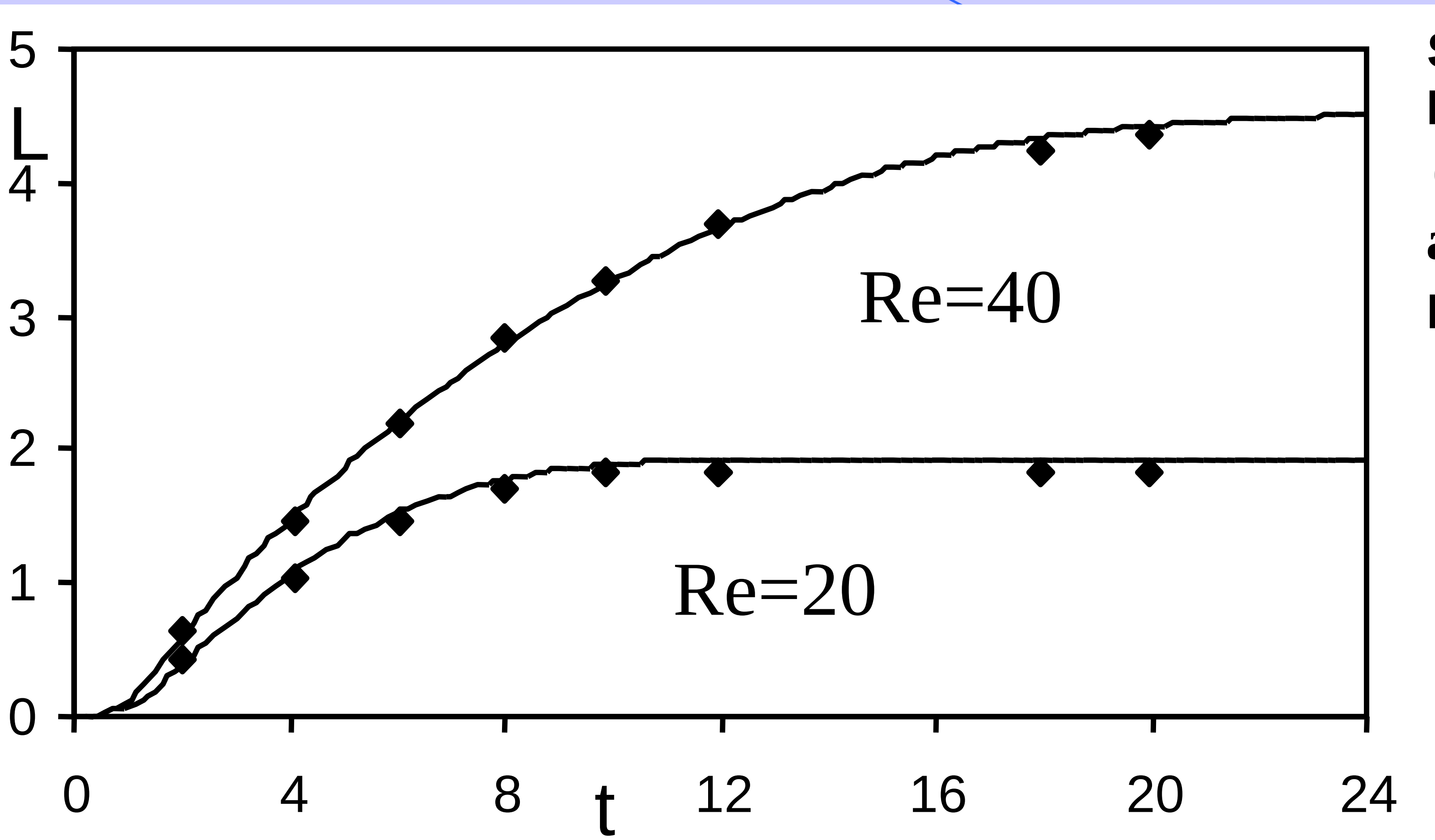


Fig.6 mesh distribution

Flow around A Circular Cylinder



Symbols-Experimental (Coutanceau et al. 1982)

Lines-Present

Fig.7 Flow at *Re* = 20
(Time evolution of the wake length)

Flow around A Circular Cylinder

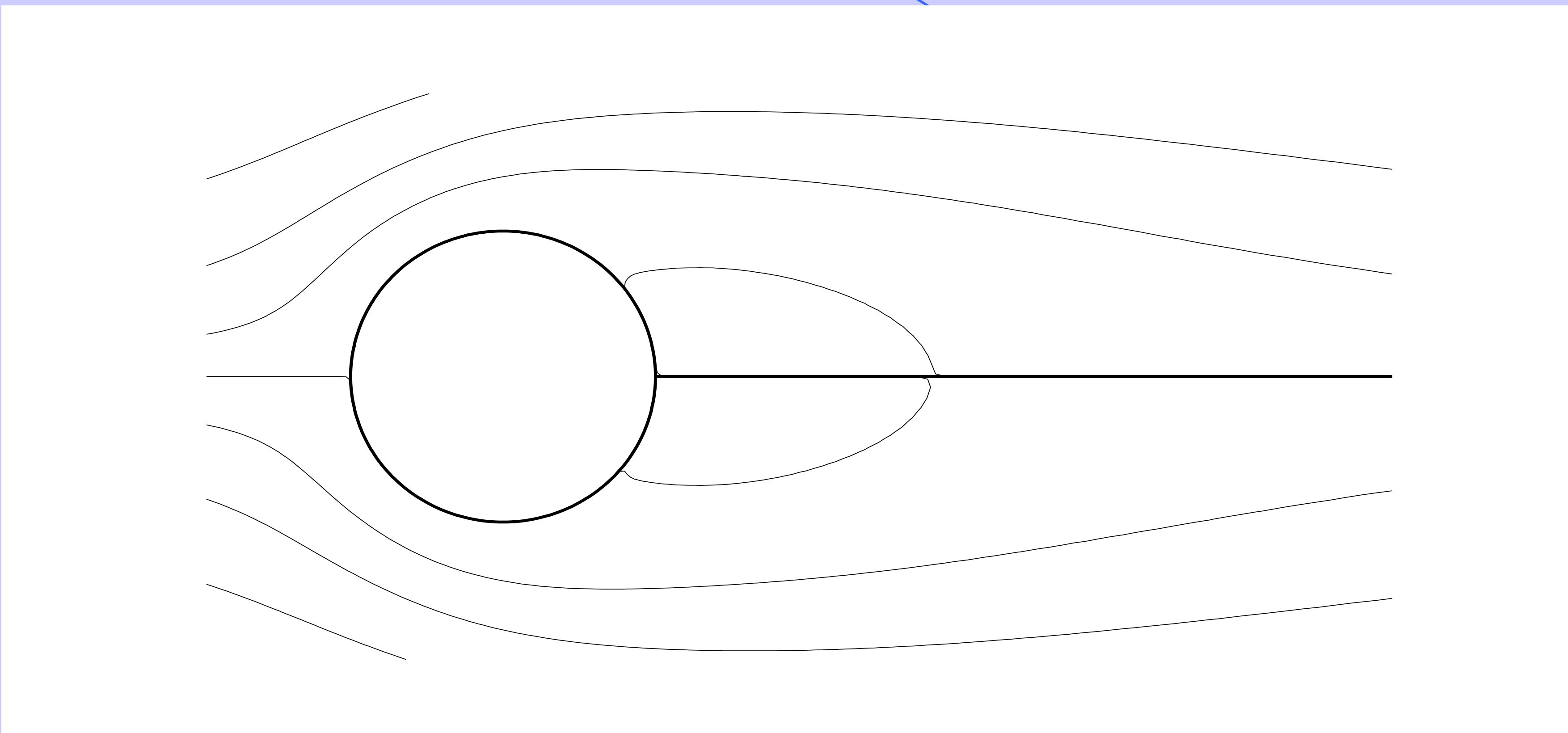


Fig.8 Flow at *Re* = 20 (streamlines)

Flow around A Circular Cylinder

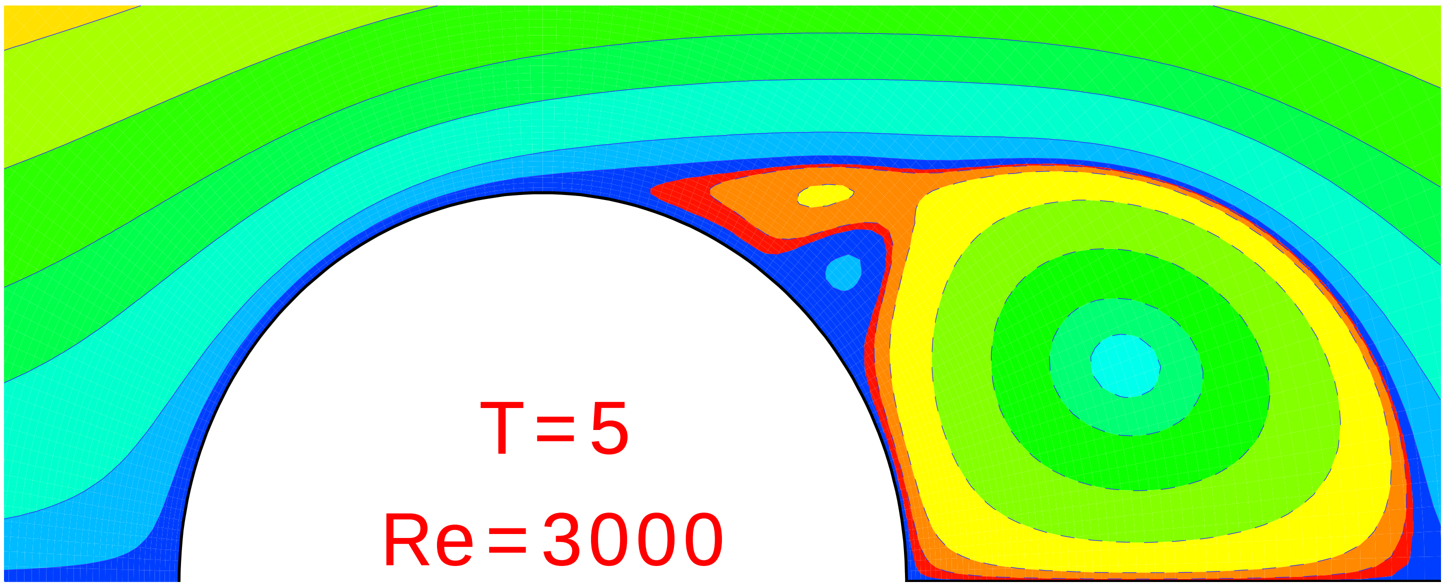


Fig. 9 Flow at Early stage at *Re* = 3000 (streamline)

Flow around A Circular Cylinder

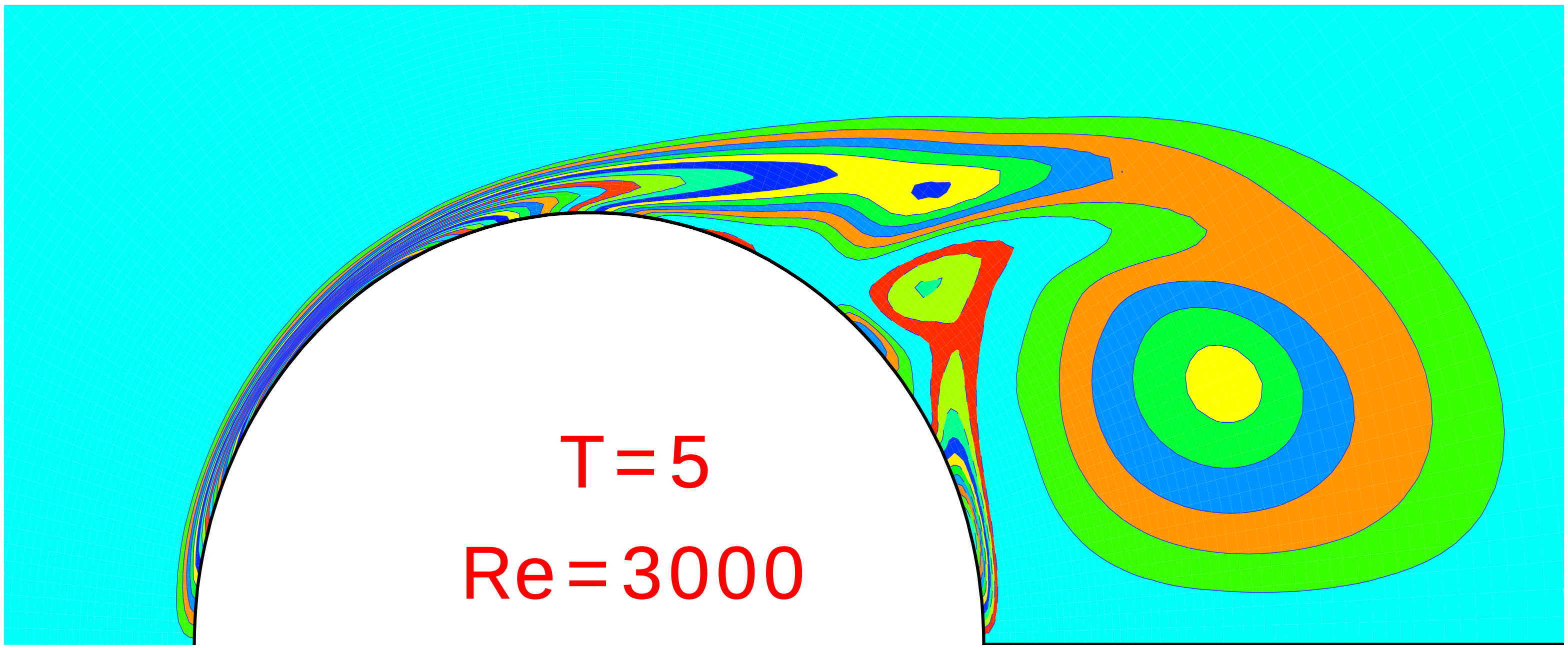
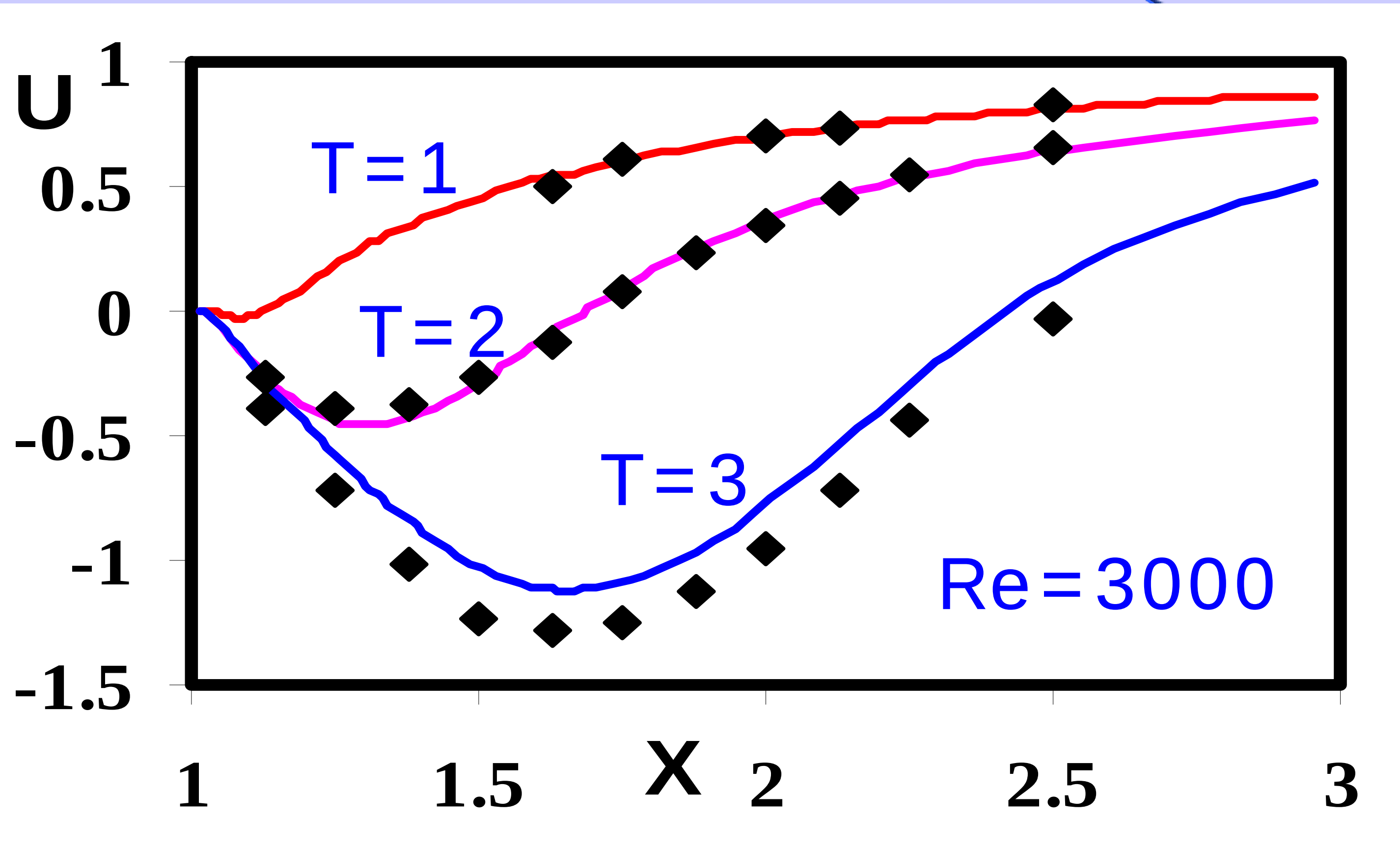


Fig.10 Flow at Early stage at *Re* = 3000 (Vorticity)

Flow around A Circular Cylinder

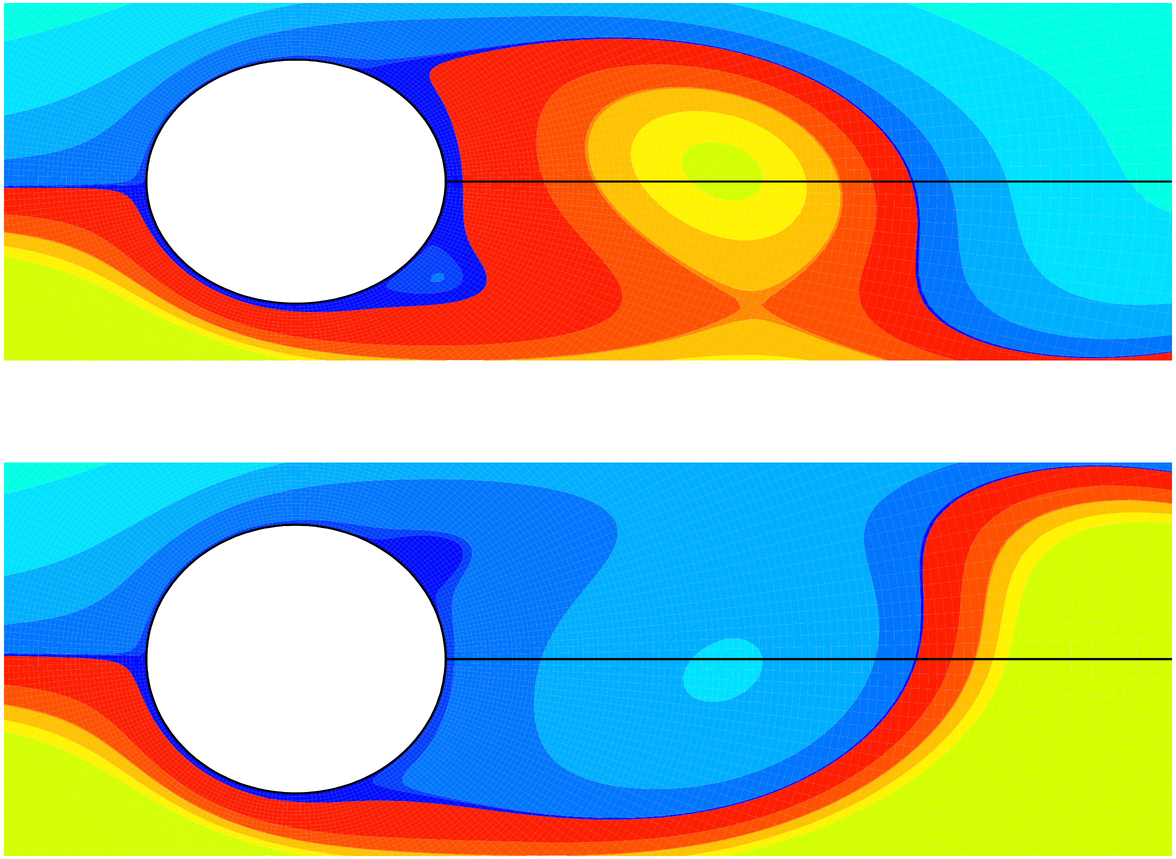


Symbols-Experimental (Bouard &Coutanceau)

Lines-Present

Fig.11 Flow at Early stage at *Re* = 3000 (Radial Velocity Distribution along Cut Line)

Flow around A Circular Cylinder



t = 3T/8

t = 7T/8

Fig. 12 Vortex Shedding (Re=100)

Natural Convection in An Annulus

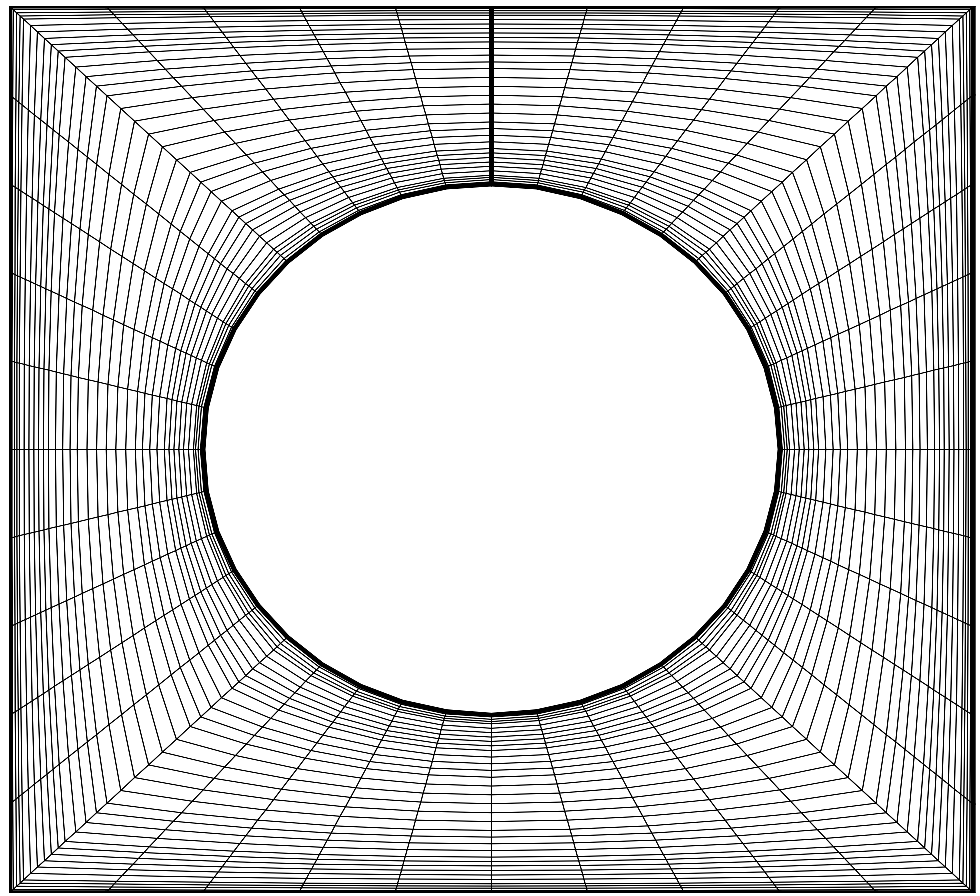


Fig. 13 Mseh in Annulus

Natural Convection in An Annulus

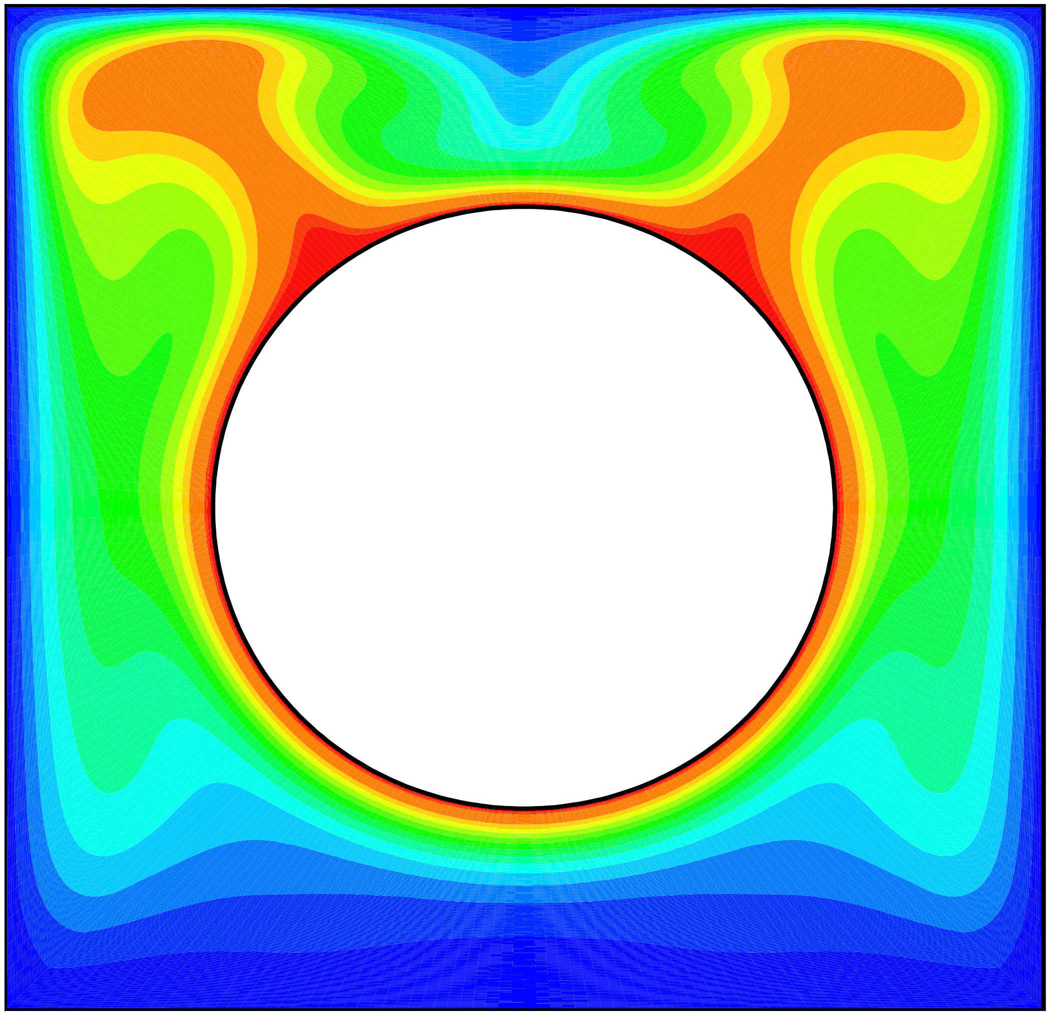


Fig. 14 Temperature Pattern

5. Conclusions

- Features of TLLBM
  - Explicit form
  - Mesh free
  - Second Order of accuracy
  - Removal of the limitation of the standard LBM
- Great potential in practical application
- Require large memory for 3D problem

  Parallel computation